

Sums of powers of Fibonacci and Lucas polynomials in terms of Fibopolynomials

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Abstract

We consider sums of powers of Fibonacci and Lucas polynomials of the form $\sum_{n=0}^q F_{tsn}^k(x)$ and $\sum_{n=0}^q L_{tsn}^k(x)$, where s, t, k are given natural numbers, together with the corresponding alternating sums $\sum_{n=0}^q (-1)^n F_{tsn}^k(x)$ and $\sum_{n=0}^q (-1)^n L_{tsn}^k(x)$. We give necessary and sufficient conditions on the parameters s, t, k for express these sums as linear combinations of the s -Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m = 1, 2, \dots, tk$. We also obtain as corollaries some results of this kind for sums of powers of Gibonacci polynomials $\sum_{n=0}^q G_{tsn}^k(x)$.

1 Introduction

We use \mathbb{N} for the natural numbers and \mathbb{N}' for $\mathbb{N} \cup \{0\}$. Throughout this work, s, t and k will denote natural numbers. We will use also the abbreviations: ‘e’ for *even*, ‘o’ for *odd*, ‘e₁’ for *even* $\equiv 2 \pmod{4}$ and ‘e₂’ for *even* $\equiv 0 \pmod{4}$.

We use the standard notation $F_n(x)$ and $L_n(x)$ for the sequences of Fibonacci and Lucas polynomials, defined by the recurrences (for $n \in \mathbb{N}$) $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$, $F_0(x) = 0$, $F_1(x) = 1$, and $L_{n+2}(x) = xL_{n+1}(x) + L_n(x)$, $L_0(x) = 2$, $L_1(x) = x$, respectively, and extended to $n \in \mathbb{Z}$ as $F_{-n}(x) = (-1)^{-n+1} F_n(x)$ and $L_{-n}(x) = (-1)^{-n} L_n(x)$. For $n \in \mathbb{N}$, polynomials $F_n(x)$ and $L_n(x)$ are monic, with $\deg F_n(x) = n - 1$ and $\deg L_n(x) = n$. Plainly we have $F_n(1) = F_n$ and $L_n(1) = L_n$, the Fibonacci and Lucas number sequences. For positive indices, the first Fibonacci polynomials are $F_2(x) = x$, $F_3(x) = x^2 + 1$, $F_4(x) = x^3 + 2x$, $F_5(x) = x^4 + 3x^2 + 1, \dots$, and the first Lucas polynomials are $L_2(x) = x^2 + 2$, $L_3(x) = x^3 + 3x$, $L_4(x) = x^4 + 4x^2 + 2$, $L_5(x) = x^5 + 5x^3 + 5x, \dots$. The basic facts we will use on Fibonacci and Lucas numbers and polynomials, are contained in the popular references [3] and [11].

We will be using extensively Binet’s formulas (without further comments):

$$F_n(x) = \frac{1}{\sqrt{x^2 + 4}} (\alpha^n(x) - \beta^n(x)) \quad \text{and} \quad L_n(x) = \alpha^n(x) + \beta^n(x), \quad (1)$$

where

$$\alpha(x) = \frac{1}{2} (x + \sqrt{x^2 + 4}) \quad \text{and} \quad \beta(x) = \frac{1}{2} (x - \sqrt{x^2 + 4}), \quad (2)$$

together with the basic relations between $\alpha(x)$ and $\beta(x)$, namely $\alpha(x) + \beta(x) = x$ and $\alpha(x)\beta(x) = -1$.

We will consider also the Generalized Fibonacci (Gibonacci) polynomial sequences $G_n(x)$, defined by the recurrence $G_{n+2}(x) = xG_{n+1}(x) + G_n(x)$, with arbitrary initial conditions $G_0(x)$ and $G_1(x)$. It is easy to see that

$$G_n(x) = G_0(x) F_{n-1}(x) + G_1(x) F_n(x). \quad (3)$$

Moreover, for any given $\eta \in \mathbb{Z}$, we have the Binet’s formula

$$G_{n+\eta}(x) = \frac{1}{\sqrt{x^2+4}} (c_1(x) \alpha^n(x) - c_2(x) \beta^n(x)), \quad (4)$$

where $c_1(x) = (G_1(x) \alpha(x) + G_0(x)) \alpha^{\eta-1}(x)$ and $c_2(x) = (G_1(x) \beta(x) + G_0(x)) \beta^{\eta-1}(x)$. Plainly, the particular cases $G_0(x) = 0$, $G_1(x) = 1$, and $G_0(x) = 2$, $G_1(x) = x$, correspond to the Fibonacci and Lucas cases, respectively.

We give now a short list of identities involving Fibonacci and Lucas polynomials, to be used (with no additional comments) in the proofs of the results presented in this work, and in simplifications of some of the given examples:

For $p \in \mathbb{N}$ we have

$$\frac{F_{(2p-1)s}(x)}{F_s(x)} = \sum_{k=0}^{p-1} (-1)^{sk} L_{2(p-k-1)s}(x) - (-1)^{s(p-1)}. \quad (5)$$

$$\frac{F_{2ps}(x)}{F_s(x)} = \sum_{k=0}^{p-1} (-1)^{sk} L_{(2p-2k-1)s}(x). \quad (6)$$

(The proofs of (5) and (6) are easy exercises —by using Binet's formulas—, left to the reader.) Some particular cases of these identities are the following

$$\frac{F_{2s}(x)}{F_s(x)} = L_s(x). \quad (7)$$

$$\frac{F_{3s}(x)}{F_s(x)} = L_{2s}(x) + (-1)^s = L_s^2(x) - (-1)^s. \quad (8)$$

$$\frac{F_{4s}(x)}{F_s(x)} = L_{3s}(x) + (-1)^s L_s(x) = L_s(x) (L_s^2(x) - 2(-1)^s). \quad (9)$$

$$\frac{F_{5s}(x)}{F_s(x)} = L_{4s}(x) + (-1)^s L_{2s}(x) + 1 = L_s^4(x, y) - 3(-1)^s L_s^2(x) + 1. \quad (10)$$

For $a, b, c, d, r \in \mathbb{Z}$ such that $a + b = c + d$, we have the so-called “index-reduction formula”:

$$F_a(x) F_b(x) - F_c(x) F_d(x) = (-1)^r (F_{a-r}(x) F_{b-r}(x) - F_{c-r}(x) F_{d-r}(x)). \quad (11)$$

(See [2], where the case $x = 1$ is discussed.) A popular version of (11) is obtained by setting $a = M$, $b = N$, $c = M + K$, $d = r = N - K$, where $M, N, K \in \mathbb{Z}$. That is

$$F_M(x) F_N(x) - F_{M+K}(x) F_{N-K}(x) = (-1)^{N-K} F_{M+K-N}(x) F_K(x). \quad (12)$$

For a given Fibonacci polynomial sequence $F_n(x) = (F_0(x), F_1(x), F_2(x), \dots)$, the *s-Fibonacci polynomial sequence* $F_{sn}(x)$ is $F_{sn}(x) = (F_0(x), F_s(x), F_{2s}(x), \dots)$, and the *s-Fibonacci polynomial factorial* of $F_{sn}(x)$, denoted by $(F_n(x))_s$, is $(F_n(x))_s = F_{sn}(x) F_{s(n-1)}(x) \cdots F_s(x)$. Given $n \in \mathbb{N}'$ and $k \in \{0, 1, \dots, n\}$, the *s-Fibopolynomial* (or *s-Fibopolynomial coefficient*), denoted by $\binom{n}{k}_{F_s(x)}$, is defined by $\binom{n}{0}_{F_s(x)} = \binom{n}{n}_{F_s(x)} = 1$, and

$$\binom{n}{k}_{F_s(x)} = \frac{(F_n(x))_s}{(F_k(x))_s (F_{n-k}(x))_s}, \quad k = 1, 2, \dots, n-1. \quad (13)$$

That is, for $k \in \{1, 2, \dots, n-1\}$ we have that

$$\binom{n}{k}_{F_s(x)} = \frac{F_{sn}(x) F_{s(n-1)}(x) \cdots F_{s(n-k+1)}(x)}{F_s(x) F_{2s}(x) \cdots F_{ks}(x)}. \quad (14)$$

Clearly we have symmetry for s -Fibopolynomials

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was the initial motivation for this work. We will see that (17) is just a particular case of the following general polynomial identities ((94) in section 4)

$$(-1)^{(s+1)q} L_s(x) \sum_{n=0}^q (-1)^{(s+1)n} F_{sn}^2(x) = (-1)^{sq} F_s(x) \sum_{n=0}^q (-1)^{sn} F_{2sn}(x) = F_{s(q+1)}(x) F_{sq}(x).$$

In this work we obtain necessary and sufficient conditions for the polynomial sums $\sum_{n=0}^q F_{tsn}^k(x)$, $\sum_{n=0}^q L_{tsn}^k(x)$, $\sum_{n=0}^q (-1)^n F_{tsn}^k(x)$ and $\sum_{n=0}^q (-1)^n L_{tsn}^k(x)$, where t, k, s are given natural numbers, can be expressed as linear combinations of the s -Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m = 1, 2, \dots, tk$. (There are certainly lots of works pursuing closed formulas for sums of powers of Fibonacci and Lucas sequences: see for example [4], [5], [6], among many others.) In section 2 we recall some facts about Z transform, which is the main tool used here and in previous results we use in this article, also recalled in section 2. The main results are presented in sections 3 and 4. Propositions 1 and 2 in section 3 contain, respectively, equivalent conditions on the positive integers t, k, s for the sums of powers $\sum_{n=0}^q F_{tsn}^k(x)$ and $\sum_{n=0}^q L_{tsn}^k(x)$ can be written as linear combinations of certain s -Fibopolynomials, and propositions 3 and 5 in section 4 contain, respectively, equivalent conditions on the positive integers t, k, s for the alternating sums of powers $\sum_{n=0}^q (-1)^n F_{tsn}^k(x)$ and $\sum_{n=0}^q (-1)^n L_{tsn}^k(x)$ can be written as linear combinations of the mentioned s -Fibopolynomials. Surprisingly, there are some intersections on the conditions on t and k in propositions 1 and 3 (and also in propositions 2 and 5), allowing us to write results for sums of the form $\sum_{n=0}^q (-1)^{sn} F_{tsn}^k(x)$ or $\sum_{n=0}^q (-1)^{(s+1)n} F_{tsn}^k(x)$ (and similar sums for Lucas polynomials), that work at the same time for sums $\sum_{n=0}^q F_{tsn}^k(x)$ and alternating sums $\sum_{n=0}^q (-1)^n F_{tsn}^k(x)$ as well, depending on the parity of s . These results are presented in section 4, corollaries 4 (for the Fibonacci case) and 6 (for the Lucas case). In section 5 we continue exploring intersections among the results obtained in sections 3 and 4: the fact that some conditions on the parameters t, k, s are the same for the Fibonacci case and for the Lucas case as well, suggests that such conditions should work for express a sum of powers of any Gibonacci polynomial sequence as linear combination of s -Fibopolynomials. This is true and we show in section 5 two examples (propositions 7 and 8). Finally, in section 6 we show some examples of the “discarded cases” of sections 3 and 4. It turns out that some of these ‘bad results’ are interesting identities, and in this section we show some of them. At the end of section 6 we show also some examples of identities obtained as derivatives of previous results.

2 Preliminaries

The Z transform maps complex sequences $a_n = (a_n)_{n=0}^\infty$ into complex (holomorphic) functions $A : U \subset \mathbb{C} \rightarrow \mathbb{C}$, defined by the Laurent series $A(z) = \sum_{n=0}^\infty a_n z^{-n}$ (also written as $\mathcal{Z}(a_n)$, defined outside the closure of the disk of convergence of the Taylor series $\sum_{n=0}^\infty a_n z^n$). We also write $a_n = \mathcal{Z}^{-1}(A(z))$ and we say the the sequence a_n is the inverse Z transform of $A(z)$. Some basic facts about the Z transform are:

- (a) \mathcal{Z} is linear and injective (same properties for \mathcal{Z}^{-1}).
- (b) If λ is a given complex number, $\lambda \neq 0$, the Z transform of the sequence λ^n is

$$\mathcal{Z}(\lambda^n) = \frac{z}{z - \lambda}. \quad (18)$$

- (c) If a_n is a sequence with Z transform $A(z)$, then the Z transform of the sequence $\lambda^n a_n$ is

$$\mathcal{Z}(\lambda^n a_n) = A\left(\frac{z}{\lambda}\right). \quad (19)$$

In particular we have

$$\mathcal{Z}((-1)^n a_n) = A(-z). \quad (20)$$

- (d) If a_n is a sequence with Z transform $A(z)$, then the Z transform of the sequence na_n is

$$\mathcal{Z}(na_n) = -z \frac{d}{dz} A(z). \quad (21)$$

(e) The Z transform of the convolution $a_n * b_n$ of two given sequences a_n and b_n (defined as $a_n * b_n = \sum_{t=0}^n a_t b_{n-t}$) is

$$\mathcal{Z}(a_n * b_n) = \mathcal{Z}(a_n) \mathcal{Z}(b_n). \quad (22)$$

For example, for given $t, k \in \mathbb{N}'$ we can write the sequence $F_{tsn}^k(x)$ as

$$F_{tsn}^k(x) = \left(\frac{1}{\sqrt{x^2 + 4}} (\alpha^{ts}(x) - \beta^{ts}(x)) \right)^k = \frac{1}{(x^2 + 4)^{\frac{k}{2}}} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \left(\alpha^{tsl}(x) \beta^{ts(k-l)}(x) \right)^n. \quad (23)$$

The linearity of \mathcal{Z} and (18) give us

$$\mathcal{Z}(F_{tsn}^k(x)) = \frac{1}{(x^2 + 4)^{\frac{k}{2}}} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \frac{z}{z - \alpha^{tsl}(x) \beta^{ts(k-l)}(x)}. \quad (24)$$

Similarly, since the sequence $L_{tsn}^k(x)$ can be expressed as

$$L_{tsn}^k(x) = (\alpha^{ts}(x) + \beta^{ts}(x))^k = \sum_{l=0}^k \binom{k}{l} \left(\alpha^{tsl}(x) \beta^{ts(k-l)}(x) \right)^n, \quad (25)$$

we have that

$$\mathcal{Z}(L_{tsn}^k(x)) = \sum_{l=0}^k \binom{k}{l} \frac{z}{z - \alpha^{tsl}(x) \beta^{ts(k-l)}(x)}. \quad (26)$$

In a recent work [10] we proved that expressions (24) and (26) can be written in a special form. The result is that (24) can be written as

$$\mathcal{Z}(F_{tsn}^k(x)) = z \frac{\sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x) z^{tk-i}}{\sum_{i=0}^{tk+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{tk+1}{i}_{F_s(x)} z^{tk+1-i}}, \quad (27)$$

and (26) can be written as

$$\mathcal{Z}(L_{tsn}^k(x)) = z \frac{\sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} L_{ts(i-j)}^k(x) z^{tk-i}}{\sum_{i=0}^{tk+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{tk+1}{i}_{F_s(x)} z^{tk+1-i}}. \quad (28)$$

From these formulas we obtained that the sequences $F_{tsn}^k(x)$ and $L_{tsn}^k(x)$ can be expressed as linear combinations of the s -Fibopolynomials $\binom{n+tk-i}{tk}_{F_s(x)}$, $i = 0, 1, \dots, tk$, according to

$$F_{tsn}^k(x) = (-1)^{s+1} \sum_{i=1}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x) \binom{n+tk-i}{tk}_{F_s(x)}, \quad (29)$$

and

$$L_{tsn}^k(x) = (-1)^{s+1} \sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} L_{ts(i-j)}^k(x) \binom{n+tk-i}{tk}_{F_s(x)}. \quad (30)$$

The denominator in (27) (or (28)) is a $(tk+1)$ -th degree z -polynomial, which we denote as $D_{s,tk+1}(x, z)$, that can be factored as

$$\sum_{i=0}^{tk+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{tk+1}{i}_{F_s(x)} z^{tk+1-i} = (-1)^{s+1} \prod_{j=0}^{tk} \left(z - \alpha^{sj}(x) \beta^{s(tk-j)}(x) \right). \quad (31)$$

(See proposition 1 in [10].) Moreover, if tk is even, $tk = 2p$ say, then we have the factorization

$$D_{s,2p+1}(x; z) = (-1)^{s+1} (z - (-1)^{sp}) \prod_{j=0}^{p-1} \left(z^2 - (-1)^{sj} L_{2s(p-j)}(x) z + 1 \right). \quad (32)$$

and if tk is odd, $tk = 2p - 1$ say, then we have the factorization

$$D_{s,2p}(x; z) = (-1)^{s+1} \prod_{j=0}^{p-1} \left(z^2 - (-1)^{sj} L_{s(2p-1-2j)}(x) z + (-1)^{(2p-1)s} \right), \quad (33)$$

(See (40) and (41) in [10].)

From (27) and (28) we see in particular that

$$\mathcal{Z}(F_n(x)) = \frac{z}{z^2 - xz - 1}, \quad (34)$$

and

$$\mathcal{Z}(L_n(x)) = \frac{z(2z - x)}{z^2 - xz - 1}, \quad (35)$$

respectively. Observe that, on one hand we have

$$\mathcal{Z}\left(\frac{d}{dx}F_n(x)\right) = \frac{d}{dx}\left(\frac{z}{z^2 - xz - 1}\right) = \frac{z^2}{(z^2 - xz - 1)^2} = (\mathcal{Z}(F_n(x)))^2. \quad (36)$$

On the other hand, by using (21), we have

$$\mathcal{Z}(nL_n(x) - xF_n(x)) = -z \frac{d}{dz} \left(\frac{z(2z - x)}{z^2 - xz - 1} \right) - \frac{xz}{z^2 - xz - 1} = \frac{(x^2 + 4)z^2}{(z^2 - xz - 1)^2}, \quad (37)$$

Thus, from (22), (36) and (37), we see that

$$\frac{d}{dx}F_n(x) = F_n(x) * F_n(x) = \frac{1}{x^2 + 4} (nL_n(x) - xF_n(x)). \quad (38)$$

Similarly, the known formula

$$\frac{d}{dx}L_n(x) = nF_n(x), \quad (39)$$

can be proved (by using the Z transform) as

$$\mathcal{Z}\left(\frac{d}{dx}L_n(x)\right) = \frac{d}{dx}\left(\frac{z(2z - x)}{z^2 - xz - 1}\right) = \frac{z(z^2 + 1)}{(z^2 - xz - 1)^2} = -z \frac{d}{dz} \left(\frac{z}{z^2 - xz - 1} \right) = \mathcal{Z}(nF_n(x)).$$

Formulas (38) and (39) will be used at the end of section 6.

3 The main results (I)

Let us consider first the Fibonacci case. From (29) we can write the sum $\sum_{n=0}^q F_{tsn}^k(x)$ as a sum of s -Fibopolynomials in a trivial way, namely

$$\sum_{n=0}^q F_{tsn}^k(x) = (-1)^{s+1} \sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x) \sum_{n=0}^q \binom{n+tk-i}{tk}_{F_s(x)}. \quad (40)$$

The interesting point is that we can write (40) as

$$\begin{aligned} \sum_{n=0}^q F_{tsn}^k(x) &= (-1)^{s+1} \sum_{m=1}^{tk-1} \sum_{i=1}^{tk-m} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x) \binom{q+m}{tk}_{F_s(x)} \\ &\quad + (-1)^{s+1} \sum_{i=1}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x) \sum_{n=0}^q \binom{n}{tk}_{F_s(x)}. \end{aligned} \quad (41)$$

Expression (41) tells us that the sum $\sum_{n=0}^q F_{tsn}^k(x)$ can be written as a linear combination of the $(tk-1)$ s -Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m = 1, 2, \dots, tk-1$, according to

$$\sum_{n=0}^q F_{tsn}^k(x) = (-1)^{s+1} \sum_{m=1}^{tk-1} \sum_{i=1}^{tk-m} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x) \binom{q+m}{tk}_{F_s(x)}, \quad (42)$$

if and only if

$$\sum_{i=1}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x) = 0. \quad (43)$$

Observe that from (24) and (27) we can write

$$\begin{aligned} &\sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x) z^{tk-i} \\ &= \frac{1}{(x^2+4)^{\frac{k}{2}}} \left(\sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \frac{1}{z - \alpha^{lts}(x) \beta^{(k-l)ts}(x)} \right) \left(\sum_{i=0}^{tk+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{tk+1}{i}_{F_s(x)} z^{tk+1-i} \right). \end{aligned} \quad (44)$$

Let us consider the factors in parentheses of the right-hand side of (44), namely

$$\Pi_1(x, z) = \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \frac{1}{z - \alpha^{lts}(x) \beta^{(k-l)ts}(x)}, \quad (45)$$

and

$$\Pi_2(x, z) = \sum_{i=0}^{tk+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{tk+1}{i}_{F_s(x)} z^{tk+1-i}. \quad (46)$$

Clearly any of the following conditions

$$\Pi_1(x, 1) = 0, \quad (47)$$

or

$$\Pi_1(x, 1) < \infty \quad \text{and} \quad \Pi_2(x, 1) = 0. \quad (48)$$

imply (43). In the following proposition we give explicit conditions on the parameters t , k and s , equivalent to (43).

Proposition 1 *The sum $\sum_{n=1}^q F_{tsn}^k(x)$ can be written as a linear combination of the s -Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m = 1, 2, \dots, tk-1$, (according to (42)), if and only if one of the following three cases occurs:*

	t	k	s
(a)	even	odd	even
(b)	odd	$\equiv 2 \pmod{4}$	odd
(c)	$\equiv 0 \pmod{4}$	odd	any

Proof. Observe that in each one of the three considered cases the product tk is even. Then, according to (32) we can write

$$\Pi_2(x, z) = (-1)^{s+1} \left(z - (-1)^{\frac{kt s}{2}} \right) \prod_{j=0}^{\frac{tk}{2}-1} \left(z^2 - (-1)^{sj} L_{2s(\frac{tk}{2}-j)}(x) z + 1 \right). \quad (49)$$

We prove first the sufficiency of the given conditions in each of the three cases.

(a) In this case the factor $\left(z - (-1)^{\frac{kt s}{2}} \right)$ of the right-hand side of (49) is $(z - 1)$, so we have $\Pi_2(x, 1) = 0$. It remains to check that $\Pi_1(x, 1)$ is finite. By writing k as $2k - 1$ (and using that t and s are even) we can see easily that

$$\begin{aligned} \Pi_1(x, 1) &= \sum_{l=0}^{2k-1} \binom{2k-1}{l} (-1)^l \frac{1}{1 - \alpha^{lts}(x) \beta^{(2k-1-l)ts}(x)} \\ &= \sum_{l=0}^{k-1} \binom{2k-1}{l} (-1)^{l+1} \left(\frac{1}{1 - \alpha^{(2k-1-l)ts}(x) \beta^{lts}(x)} - \frac{1}{1 - \alpha^{lts}(x) \beta^{(2k-1-l)ts}(x)} \right), \end{aligned}$$

and then we can write finally

$$\Pi_1(x, 1) = \sqrt{x^2 + 4} \sum_{l=0}^{k-1} \binom{2k-1}{l} (-1)^{l+1} \frac{F_{(2k-1-2l)ts}(x)}{2 - L_{(2k-1-2l)ts}(x)}, \quad (50)$$

so we have that $\Pi_1(x, 1)$ is finite, and then the right-hand side of (44) is equal to zero when $z = 1$, as wanted.

(b) In this case the factor $\left(z - (-1)^{\frac{kt s}{2}} \right)$ of the right-hand side of (49) is $(z + 1)$, so $\Pi_2(x, 1) \neq 0$. However, by writing k as $2K$, where K is odd, and using that t and s are odd, we can see that

$$\begin{aligned} \Pi_1(x, 1) &= \sum_{l=0}^{2K} \binom{2K}{l} (-1)^l \frac{1}{1 - \alpha^{lts}(x) \beta^{(2K-l)ts}(x)} \\ &= \sum_{l=0}^{K-1} \binom{2K}{l} (-1)^l \left(\frac{1}{1 - \alpha^{lts}(x) \beta^{(2K-l)ts}(x)} + \frac{1}{1 - \alpha^{(2K-l)ts}(x) \beta^{lts}(x)} \right) - \frac{1}{2} \binom{2K}{K} \\ &= \sum_{l=0}^{K-1} \binom{2K}{l} (-1)^l - \frac{1}{2} \binom{2K}{K} \\ &= 0. \end{aligned}$$

Thus, the right-hand side of (44) is equal to 0 when $z = 1$, as wanted.

(c) In this case the factor $\left(z - (-1)^{\frac{kt s}{2}} \right)$ of the right-hand side of (49) is $(z - 1)$, so we have $\Pi_2(x, 1) = 0$. It remains to check that $\Pi_1(x, 1)$ is finite. By writing k as $2k - 1$, and using that t is multiple of 4, we can see that (50) is valid for any $s \in \mathbb{N}$. Thus the right-hand side of (44) is 0 when $z = 1$, as wanted.

This ends the proof of the sufficiency of the given conditions. We will prove now the necessity of such conditions. We have the following possibilities for the parameters t, k, s :

	t	k	s
(i)	e ₁	o	e
(ii)	e ₂	o	e
(iii)	e ₂	o	o
(iv)	o	e ₁	o
(v)	e	e	e

	t	k	s
(vi)	e	e	o
(vii)	o	e	e
(viii)	e ₁	o	o
(ix)	o	e ₂	o
(x)	o	o	e
(xi)	o	o	o

We have proved that in the cases (i), (ii), (iii) and (iv), condition (43) holds. We have to show now that in the remaining cases, condition (43) does not hold. To this end, we will show some counterexamples (we leave the details of the corresponding calculations and/or simplifications to the reader).

For (v) and (vi) take $t = k = 2$. In this case we have, for any $s \in \mathbb{N}$, that

$$\sum_{i=1}^4 \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{5}{j}_{F_s(x)} F_{2s(i-j)}^2(x) = 2F_{2s}^2(x) (L_{2s}(x) + 2(-1)^{s+1}). \quad (51)$$

For (vii) and (ix) take $t = 1, k = 4$. In this case we have, for any $s \in \mathbb{N}$, that

$$\sum_{i=1}^4 \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{5}{j}_{F_s(x)} F_{s(i-j)}^4(x) = -6F_s^4(x) (L_{2s}(x) + 2(-1)^s). \quad (52)$$

For (viii) take $t = 2, k = 1$ and s odd. In this case we have that

$$\sum_{i=1}^2 \sum_{j=0}^i (-1)^{\frac{j(j+1)}{2}} \binom{3}{j}_{F_s(x)} F_{2s(i-j)}(x) = 2F_{2s}(x) L_s(x). \quad (53)$$

For (x) and (xi) take $t = k = 1$. In this case we have, for any $s \in \mathbb{N}$, that

$$\sum_{i=1}^1 \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{2}{j}_{F_s(x)} F_{s(i-j)}(x) = (-1)^{s+1} F_s(x). \quad (54)$$

This ends our proof. ■

An example from the case (c) of proposition 1 is the following identity (corresponding to $t = 4$ and $k = 1$), valid for any $s \in \mathbb{N}$

$$\sum_{n=0}^q F_{4sn}(x) = F_{4s}(x) \left(\binom{q+1}{4}_{F_s(x)} + (-1)^{s+1} L_{2s}(x) \binom{q+2}{4}_{F_s(x)} + \binom{q+3}{4}_{F_s(x)} \right). \quad (55)$$

Let us consider now the case of sums of powers of Lucas polynomials. From (30) we see that

$$(-1)^{s+1} \sum_{n=0}^q L_{tsn}^k(x) = \sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} L_{ts(i-j)}^k(x) \sum_{n=0}^q \binom{n+tk-i}{tk}_{F_s(x)}, \quad (56)$$

which can be written as

$$\begin{aligned} \sum_{n=0}^q L_{tsn}^k(x) &= (-1)^{s+1} \sum_{m=1}^{tk} \sum_{i=0}^{tk-m} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} L_{ts(i-j)}^k(x) \binom{q+m}{tk}_{F_s(x)} \\ &\quad + (-1)^{s+1} \sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} L_{ts(i-j)}^k(x) \sum_{n=0}^q \binom{n}{tk}_{F_s(x)}. \end{aligned} \quad (57)$$

Expression (57) tells us that the sum $\sum_{n=0}^q L_{tsn}^k(x)$ can be written as a linear combination of the (tk) s -Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m = 1, 2, \dots, tk$, according to

$$\sum_{n=0}^q L_{tsn}^k(x) = (-1)^{s+1} \sum_{m=1}^{tk} \sum_{i=0}^{tk-m} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} L_{ts(i-j)}^k(x) \binom{q+m}{tk}_{F_s(x)}, \quad (58)$$

if and only if

$$\sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} L_{ts(i-j)}^k(x) = 0. \quad (59)$$

From (26) and (28) we can write

$$\begin{aligned} & \sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} L_{ts(i-j)}^k(x) z^{tk-i} \\ &= \left(\sum_{i=0}^{tk+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{tk+1}{i}_{F_s(x)} z^{tk+1-i} \right) \sum_{l=0}^k \binom{k}{l} \frac{1}{z - \alpha^{lts}(x) \beta^{(k-l)ts}(x)}. \end{aligned} \quad (60)$$

We have again the factor $\Pi_2(x, z)$ considered in the Fibonacci case (see (46)), and the factor

$$\tilde{\Pi}_1(x, z) = \sum_{l=0}^k \binom{k}{l} \frac{1}{z - \alpha^{lts}(x) \beta^{(k-l)ts}(x)}. \quad (61)$$

Plainly any of the conditions: (a) $\tilde{\Pi}_1(x, 1) = 0$, or, (b) $\tilde{\Pi}_1(x, 1) < \infty$ and $\Pi_2(x, 1) = 0$, imply (59). In the following result we show explicit conditions on t , k and s , equivalent to (59).

Proposition 2 *The sum $\sum_{n=0}^q L_{tsn}^k(x)$ can be written as a linear combination of the s -Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m = 1, 2, \dots, tk$, according to (45), if and only if one of the following two cases occurs:*

	t	k	s
(a)	even	odd	even
(b)	$\equiv 0 \pmod{4}$	odd	any

Proof. In both cases we have tk even, so it is valid the factorization (49) for $\Pi_2(x, z)$. Let us consider each of the cases.

(a) In this case the factor $(z - (-1)^{\frac{kts}{2}})$ in $\Pi_2(x, z)$ is $(z - 1)$, so we have $\Pi_2(x, 1) = 0$. It remains to check that $\tilde{\Pi}_1(x, 1)$ is finite. In fact, by writing k as $2k - 1$ and using that t and s are even, we can see that

$$\begin{aligned} \tilde{\Pi}_1(x, 1) &= \sum_{l=0}^{2k-1} \binom{2k-1}{l} \frac{1}{1 - \alpha^{lts}(x) \beta^{(2k-1-l)ts}(x)} \\ &= \sum_{l=0}^{k-1} \binom{2k-1}{l} \left(\frac{1}{1 - \alpha^{lts}(x) \beta^{(2k-1-l)ts}(x)} + \frac{1}{1 - \alpha^{(2k-1-l)ts}(x) \beta^{lts}(x)} \right) \\ &= \sum_{l=0}^{k-1} \binom{2k-1}{l} \\ &= 4^{k-1}. \end{aligned}$$

(b) In this case the factor $(z - (-1)^{\frac{kts}{2}})$ in $\Pi_2(x, z)$ is again $(z - 1)$, so we have $\Pi_2(x, 1) = 0$. With a similar calculation to the case (a), we can see that in this case we have also $\tilde{\Pi}_1(x, 1) = 4^{k-1}$.

This ends the proof of the sufficiency of the given conditions. Let us prove now that they are necessary.

We have the following possibilities for the parameters t, k, s :

	t	k	s
(i)	e ₁	o	e
(ii)	e ₂	o	e
(iii)	e ₂	o	o
(iv)	e	e	e

	t	k	s
(v)	e	e	o
(vi)	o	e	e
(vii)	e ₁	o	o
(viii)	o	e	o
(ix)	o	o	e
(x)	o	o	o

We have proved that (59) holds in the cases (i), (ii) and (iii). Let us see, by means of counterexamples, that in the remaining cases (59) does not hold.

If $t = k = 2$, we have for any $s \in \mathbb{N}$ the following identity (corresponding to cases (iv) and (v))

$$\sum_{i=0}^4 \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{5}{j}_{F_s(x)} L_{2s(i-j)}^2(x) = -2(x^2 + 4)^2 L_s^2(x) F_s^4(x). \quad (62)$$

If $t = 1$ and $k = 2$, we have for any $s \in \mathbb{N}$ the following identity (corresponding to cases (vi) and (viii))

$$\sum_{i=0}^2 \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{3}{j}_{F_s(x)} L_{s(i-j)}^2(x) = (3(-1)^s - 1)(L_{2s}(x) - 2). \quad (63)$$

If $t = 2$ and $k = 1$, we have for s odd the following identity (corresponding to case (vii))

$$\sum_{i=0}^2 \sum_{j=0}^i (-1)^{\frac{j(j+1)}{2}} \binom{3}{j}_{F_s(x)} L_{2s(i-j)}(x) = 4 - 2L_{2s}(x). \quad (64)$$

If $t = k = 1$, we have for any $s \in \mathbb{N}$ the following identity (corresponding to cases (ix) and (x))

$$\sum_{i=0}^1 \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{2}{j}_{F_s(x)} L_{s(i-j)}(x) = (-1)^s (L_s(x) - 2). \quad (65)$$

The proof is now complete. ■

An example from the case (b) of proposition 2 is the following identity (corresponding to $t = 4$ and $k = 1$), valid for any $s \in \mathbb{N}$

$$\begin{aligned} \sum_{n=0}^q L_{4sn}(x) &= 2 \binom{q+4}{4}_{F_s(x)} + (-L_{4s}(x) + 2(-1)^{s+1} L_{2s}(x)) \binom{q+3}{4}_{F_s(x)} \\ &\quad + (-1)^s (L_{6s}(x) + L_{2s}(x) + 2(-1)^s) \binom{q+2}{4}_{F_s(x)} - L_{4s}(x) \binom{q+1}{4}_{F_s(x)}. \end{aligned} \quad (66)$$

Examples from the cases (a) and (b) of proposition 1, and from the case (a) of proposition 2, will be given in section 4, since some variants of them work also as examples of alternating sums of powers of Fibonacci or Lucas polynomials, to be discussed in section 4 (see corollaries 4 and 6).

4 The main results (II): alternating sums

According to (20), (24), (26), (27) and (28), the Z transform of the alternating sequences $(-1)^n F_{tsn}^k(x)$ and $(-1)^n L_{tsn}^k(x)$ are

$$\begin{aligned} \mathcal{Z}((-1)^n F_{tsn}^k(x)) &= \frac{1}{(x^2+4)^{\frac{k}{2}}} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \frac{z}{z + \alpha^{lts}(x) \beta^{(k-l)ts}(x)} \\ &= -z \frac{\sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x) (-z)^{tk-i}}{\sum_{i=0}^{tk+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{tk+1}{i}_{F_s(x)} (-z)^{tk+1-i}}, \end{aligned} \quad (67)$$

and

$$\begin{aligned} \mathcal{Z}((-1)^n L_{tsn}^k(x)) &= \sum_{l=0}^k \binom{k}{l} \frac{z}{z + \alpha^{lts}(x) \beta^{(k-l)ts}(x)} \\ &= -z \frac{\sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} L_{ts(i-j)}^k(x) (-z)^{tk-i}}{\sum_{i=0}^{tk+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{tk+1}{i}_{F_s(x)} (-z)^{tk+1-i}}. \end{aligned} \quad (68)$$

By using (29) and (30) it is possible to establish expressions, for the case of alternating sums, similar to expressions (41) and (57), namely

$$\begin{aligned} &\sum_{n=0}^q (-1)^n F_{tsn}^k(x) \\ &= (-1)^{s+1} \sum_{m=1}^{tk-1} \sum_{i=1}^{tk-m} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2} + i + tk + q + m} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x) \binom{q+m}{tk}_{F_s(x)} \\ &\quad + (-1)^{s+1} \sum_{i=1}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2} + i + tk} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x) \sum_{n=0}^q (-1)^n \binom{n}{tk}_{F_s(x)}, \end{aligned} \quad (69)$$

and

$$\begin{aligned} &\sum_{n=0}^q (-1)^n L_{tsn}^k(x) \\ &= (-1)^{s+1} \sum_{m=1}^{tk} \sum_{i=0}^{tk-m} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2} + i + tk + q + m} \binom{tk+1}{j}_{F_s(x)} L_{ts(i-j)}^k(x) \binom{q+m}{tk}_{F_s(x)} \\ &\quad + (-1)^{s+1} \sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2} + i + tk} \binom{tk+1}{j}_{F_s(x)} L_{ts(i-j)}^k(x) \sum_{n=0}^q (-1)^n \binom{n}{tk}_{F_s(x)}, \end{aligned} \quad (70)$$

respectively. From (69) and (70) we see that the alternating sums of powers $\sum_{n=0}^q (-1)^n F_{tsn}^k(x)$ and $\sum_{n=0}^q (-1)^n L_{tsn}^k(x)$ can be written as linear combinations of s -Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$ according to

$$\begin{aligned} &\sum_{n=0}^q (-1)^n F_{tsn}^k(x) \\ &= (-1)^{s+1} \sum_{m=1}^{tk-1} \sum_{i=1}^{tk-m} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2} + i + tk + q + m} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x) \binom{q+m}{tk}_{F_s(x)}, \end{aligned} \quad (71)$$

and

$$\begin{aligned} & \sum_{n=0}^q (-1)^n L_{tsn}^k(x) \\ = & (-1)^{s+1} \sum_{m=1}^{tk} \sum_{i=0}^{tk-m} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i+tk+q+m} \binom{tk+1}{j}_{F_s(x)} L_{ts(i-j)}^k(x) \binom{q+m}{tk}_{F_s(x)}, \end{aligned} \quad (72)$$

if and only if we have that

$$\sum_{i=1}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x) = 0, \quad (73)$$

and

$$\sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i} \binom{tk+1}{j}_{F_s(x)} L_{ts(i-j)}^k(x) = 0, \quad (74)$$

respectively. Observe that, according to (67) and (68) we have in these cases that

$$\begin{aligned} & (x^2 + 4)^{\frac{k}{2}} \sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x) z^{tk-i} \\ = & \left(\sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \frac{1}{z + \alpha^{lts}(x) \beta^{(k-l)ts}(x)} \right) \left(\sum_{i=0}^{tk+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}+i} \binom{tk+1}{i}_{F_s(x)} z^{tk+1-i} \right), \end{aligned} \quad (75)$$

and

$$\begin{aligned} & \sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i} \binom{tk+1}{j}_{F_s(x)} L_{ts(i-j)}^k(x) z^{tk-i} \\ = & \left(\sum_{l=0}^k \binom{k}{l} \frac{1}{z + \alpha^{lts}(x) \beta^{(k-l)ts}(x)} \right) \left(\sum_{i=0}^{tk+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}+i} \binom{tk+1}{i}_{F_s(x)} z^{tk+1-i} \right), \end{aligned} \quad (76)$$

respectively. Then we need to consider now the following factors

$$\Omega_1(x, z) = \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \frac{1}{z + \alpha^{lts}(x) \beta^{(k-l)ts}(x)}, \quad (77)$$

$$\tilde{\Omega}_1(x, z) = \sum_{l=0}^k \binom{k}{l} \frac{1}{z + \alpha^{lts}(x) \beta^{(k-l)ts}(x)}, \quad (78)$$

and

$$\Omega_2(x, z) = \sum_{i=0}^{tk+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}+i} \binom{tk+1}{i}_{F_s(x)} z^{tk+1-i}. \quad (79)$$

Plainly (73) is concluded from any of the conditions: (a) $\Omega_1(x, 1) = 0$, or, (b) $\Omega_1(x, 1) < \infty$ and $\Omega_2(x, 1) = 0$, and (74) is concluded from any of the conditions: (a) $\tilde{\Omega}_1(x, 1) = 0$, or, (b) $\tilde{\Omega}_1(x, 1) < \infty$ and $\Omega_2(x, 1) = 0$. In this section we give conditions on the parameters t , k and s , equivalent to (73) (for the Fibonacci case: proposition 3), and equivalent to (74) (for the Lucas case: proposition 5).

In the Fibonacci case we have the following result.

Proposition 3 *The alternating sum $\sum_{n=0}^q (-1)^n F_{tsn}^k(x)$ can be written as a linear combination of the s -Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m = 1, 2, \dots, tk-1$, according to (71), if and only if one of the following cases occurs:*

	t	k	s
(a)	any	$\equiv 0 \pmod{4}$	any
(b)	any	even	even
(c)	$\equiv 2 \pmod{4}$	any	odd
(d)	even	even	any

Proof. Observe that in all the four cases we have tk even. Thus, according to (32) (with z replaced by $-z$) we can factor $\Omega_2(x, z)$ as

$$\Omega_2(x, z) = (-1)^s \left(z + (-1)^{\frac{tsk}{2}} \right) \prod_{j=0}^{\frac{tk}{2}-1} \left(z^2 + (-1)^{sj} L_{2s(\frac{tk}{2}-j)}(x) z + 1 \right). \quad (80)$$

We have to show that (73) holds in each of the four cases (and only in them).

(a) In this case the factor $\left(z + (-1)^{\frac{tsk}{2}} \right)$ of (80) is $z + 1$, so we have $\Omega_2(x, 1) \neq 0$. However, by setting $z = 1$ in (77), with k replaced by $4k$, we get

$$\begin{aligned} \Omega_1(x, 1) &= \sum_{l=0}^{4k} \binom{4k}{l} (-1)^l \frac{1}{1 + \alpha^{lts}(x) \beta^{(4k-l)ts}(x)} \\ &= \sum_{l=0}^{2k-1} \binom{4k}{l} (-1)^l \left(\frac{1}{1 + \alpha^{lts}(x) \beta^{(4k-l)ts}(x)} + \frac{1}{1 + \alpha^{(4k-l)ts}(x) \beta^{lts}(x)} \right) + \frac{1}{2} \binom{4k}{2k} \\ &= \sum_{l=0}^{2k-1} \binom{4k}{l} (-1)^l + \frac{1}{2} \binom{4k}{2k} \\ &= 0. \end{aligned}$$

Thus (73) holds, as wanted.

(b) In this case we have $z + (-1)^{\frac{tsk}{2}} = z + 1$, and then $\Omega_2(x, 1) \neq 0$. However, by setting $z = 1$ in (77), with k substituted by $2k$, we get for s even

$$\begin{aligned} \Omega_1(x, 1) &= \sum_{l=0}^{2k} \binom{2k}{l} (-1)^l \frac{1}{1 + \alpha^{2lts}(x) \beta^{2(2k-l)ts}(x)} \\ &= \sum_{l=0}^{k-1} \binom{2k}{l} (-1)^l \left(\frac{1}{1 + \alpha^{2lts}(x) \beta^{2(2k-l)ts}(x)} + \frac{1}{1 + \alpha^{2(2k-l)ts}(x) \beta^{2lts}(x)} \right) + \frac{1}{2} \binom{2k}{k} (-1)^k \\ &= \sum_{l=0}^{k-1} \binom{2k}{l} (-1)^l + \frac{1}{2} \binom{2k}{k} (-1)^k \\ &= 0. \end{aligned} \quad (81)$$

Thus (73) holds, as wanted.

(c) If in (80) we set $z = 1$ and replace t by $2(2t-1)$, we obtain that

$$\Omega_2(x, 1) = \left(1 + (-1)^k \right) \prod_{j=0}^{(2t-1)k-1} \left((-1)^j L_{2s((2t-1)k-j)}(x) + 2 \right). \quad (82)$$

We consider two subcases:

(c1) Suppose that k is even. In this case we have $\Omega_2(x, 1) \neq 0$. But with a similar calculation of (81) we can see that in this case we have also $\Omega_1(x, 1) = 0$. Thus (73) holds when k is even.

(c2) Suppose that k is odd. In this case we have clearly that $\Omega_2(x, 1) = 0$. We just need to check that $\Omega_1(x, 1)$ is finite. If we set $z = 1$ in (77) and substitute k by $2k - 1$, we obtain for $t \equiv 2 \pmod{4}$ and s odd

$$\begin{aligned}\Omega_1(x, 1) &= \sum_{l=0}^{2k-1} \binom{2k-1}{l} (-1)^{l+1} \frac{1}{1 + \alpha^{lts}(x) \beta^{(2k-1-l)ts}(x)} \\ &= \sum_{l=0}^{k-1} \binom{2k-1}{l} (-1)^{l+1} \left(\frac{1}{1 + \alpha^{lts}(x) \beta^{(2k-1-l)ts}(x)} - \frac{1}{1 + \alpha^{(2k-1-l)ts}(x) \beta^{lts}(x)} \right) \\ &= \sqrt{x^2 + 4} \sum_{l=0}^{k-1} \binom{2k-1}{l} (-1)^{l+1} \frac{F_{(2k-1-2l)ts}(x)}{2 + L_{(2k-1-2l)ts}(x)}.\end{aligned}$$

Then we have $\Omega_1(x, 1) < \infty$, as wanted. That is, expression (73) holds when k is odd.

(d) In this case the factor $\left(z + (-1)^{\frac{tsk}{2}}\right)$ of (80) is $(z + 1)$, so we have $\Omega_2(x, 1) \neq 0$. But with a similar calculation of (81) we can see that in this case we have also $\Omega_1(x, 1) = 0$. That is, in this case we have also that $\Omega_1(x, 1) = 0$, and we conclude that (73) holds.

This ends the proof of the sufficiency of the given conditions for (73) hold. In the following table we have the list of all parity possibilities for the parameters t , k and s :

	t	k	s
(i)	e	e	e
(ii)	e	e	o
(iii)	o	e	e
(iv)	e ₁	o	o
(v)	o	e ₂	o

	t	k	s
(vi)	e	o	e
(vii)	e ₂	o	o
(viii)	o	e ₁	o
(ix)	o	o	e
(x)	o	o	o

We have proved that (73) holds in cases (i) to (v). Then we have to prove that in the remaining cases expression (73) does not hold. To this end, we show some counterexamples.

For $t = 4, k = 1$ (cases (vi) and (vii)) we have for any s

$$\sum_{i=1}^4 \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i} \binom{5}{j}_{F_s(x)} F_{4s(i-j)}(x) = F_{4s}(x) (2L_{5s}(x) + 6(-1)^s L_{3s}(x) + 8L_s(x)). \quad (83)$$

For $t = 1, k = 2$ (case (viii)) we have for odd s

$$\sum_{i=1}^2 \sum_{j=0}^i (-1)^{\frac{j(j+1)}{2}+i} \binom{3}{j}_{F_s(x)} F_{s(i-j)}^2(x) = -2F_s^2(x). \quad (84)$$

For $t = k = 1$ (cases (ix) and (x)) we have for any s

$$\sum_{i=1}^1 \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i} \binom{2}{j}_{F_s(x)} F_{s(i-j)}(x) = (-1)^s F_s(x). \quad (85)$$

Our proof is complete. ■

Corollary 4 (a) If t is odd and $k \equiv 2 \pmod{4}$, we have the following identity valid for any $s \in \mathbb{N}$

$$\begin{aligned}& \sum_{n=0}^q (-1)^{(s+1)n} F_{tsn}^k(x) \\ &= (-1)^{s+1} \sum_{m=1}^{tk-1} \sum_{i=1}^{tk-m} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}+(s+1)(i+tk+q+m)} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x) \binom{q+m}{tk}_{F_s(x)}.\end{aligned} \quad (86)$$

(b) If $t \equiv 2 \pmod{4}$ and k is odd, we have the following identity valid for any $s \in \mathbb{N}$

$$\begin{aligned} & \sum_{n=0}^q (-1)^{sn} F_{tsn}^k(x) \\ &= (-1)^{s+1} \sum_{m=1}^{tk-1} \sum_{i=1}^{tk-m} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2} + s(i+tk+q+m)} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x) \binom{q+m}{tk}_{F_s(x)}. \end{aligned} \quad (87)$$

Proof. (a) When t is odd and $k \equiv 2 \pmod{4}$, formula (86) with s odd, gives the result (42) of case (b) of proposition 1 (which is valid for t odd, $k \equiv 2 \pmod{4}$ and s odd). Similarly, for t odd and $k \equiv 2 \pmod{4}$, formula (86) with s even, gives the result (71) of case (b) of proposition 3 (which is valid for even k and s , and any t).

(b) When $t \equiv 2 \pmod{4}$ and k is odd, formula (87) with s even, gives the result (42) of case (a) of proposition 1 (which is valid for t even, k odd and s even). Similarly, for $t \equiv 2 \pmod{4}$ and k odd, formula (87) with s odd, gives the result (71) of case (c) of proposition 3 (which is valid for $t \equiv 2 \pmod{4}$, s odd and any k). ■

We give some examples from the cases considered in corollary 4. Beginning with the case (a), we have the following identities, valid for $s \in \mathbb{N}$.

By setting $t = 1$ and $k = 2$ in (86) we get

$$\sum_{n=0}^q (-1)^{(s+1)(n+q)} F_{sn}^2(x) = F_s^2(x) \binom{q+1}{2}_{F_s(x)}. \quad (88)$$

By setting $t = 3$ and $k = 2$ in (86) we obtain

$$\sum_{n=0}^q (-1)^{(s+1)(n+q)} F_{3sn}^2(x) = F_{3s}^2(x) \left(\binom{q+1}{6}_{F_s(x)} + \binom{q+5}{6}_{F_s(x)} + ((-1)^{s+1} L_{4s}(x) - L_{2s}(x)) \left(\binom{q+2}{6}_{F_s(x)} + \binom{q+4}{6}_{F_s(x)} \right) + (-1)^s (L_{6s}(x) + L_s^2(x)) \binom{q+3}{6}_{F_s(x)} \right). \quad (89)$$

By setting $t = 1$ and $k = 6$ in (86) we get

$$\sum_{n=0}^q (-1)^{(s+1)(n+q)} F_{sn}^6(x) = F_s^6(x) \left(\binom{q+1}{6}_{F_s(x)} + \binom{q+5}{6}_{F_s(x)} + L_s^2(x) ((-1)^s 5L_{2s}(x) + 4) \left(\binom{q+2}{6}_{F_s(x)} + \binom{q+4}{6}_{F_s(x)} \right) + ((-1)^s 10L_{6s}(x) + 30L_{4s}(x) + (-1)^s 52L_{2s}(x) + 62) \binom{q+3}{6}_{F_s(x)} \right). \quad (90)$$

Three examples from the case (b) of corollary 4 are the following identities, valid for $s \in \mathbb{N}$.

By setting $t = 2$ and $k = 1$ in (87) we obtain

$$\sum_{n=0}^q (-1)^{s(n+q)} F_{2sn}(x) = F_{2s}(x) \binom{q+1}{2}_{F_s(x)}. \quad (91)$$

By setting $t = 2$ and $k = 3$ in (87) we obtain

$$\sum_{n=0}^q (-1)^{s(n+q)} F_{2sn}^3(x) = F_{2s}^3(x) \left(\begin{aligned} & \left(\begin{smallmatrix} q+1 \\ 6 \end{smallmatrix} \right)_{F_s(x)} + \left(\begin{smallmatrix} q+5 \\ 6 \end{smallmatrix} \right)_{F_s(x)} \\ & + \left((-1)^{s+1} L_{4s}(x) + 2L_{2s}(x) \right) \left(\left(\begin{smallmatrix} q+2 \\ 6 \end{smallmatrix} \right)_{F_s(x)} + \left(\begin{smallmatrix} q+4 \\ 6 \end{smallmatrix} \right)_{F_s(x)} \right) \\ & + 2(-1)^{s+1} \left(L_{6s}(x) + L_{2s}(x) + (-1)^{s+1} \right) \left(\begin{smallmatrix} q+3 \\ 6 \end{smallmatrix} \right)_{F_s(x)} \end{aligned} \right). \quad (92)$$

By setting $t = 6$ and $k = 1$ in (87) we get

$$\sum_{n=0}^q (-1)^{s(n+q)} F_{6sn}(x) = F_{6s}(x) \left(\begin{aligned} & \left(\begin{smallmatrix} q+1 \\ 6 \end{smallmatrix} \right)_{F_s(x)} + \left(\begin{smallmatrix} q+5 \\ 6 \end{smallmatrix} \right)_{F_s(x)} \\ & + \left((-1)^{s+1} L_{4s}(x) - L_{2s}(x) \right) \left(\left(\begin{smallmatrix} q+2 \\ 6 \end{smallmatrix} \right)_{F_s(x)} + \left(\begin{smallmatrix} q+4 \\ 6 \end{smallmatrix} \right)_{F_s(x)} \right) \\ & + (-1)^s \left(L_{6s}(x) + L_s^2(x) \right) \left(\begin{smallmatrix} q+3 \\ 6 \end{smallmatrix} \right)_{F_s(x)} \end{aligned} \right). \quad (93)$$

From (88) and (91) we see that

$$(-1)^{(s+1)q} L_s(x) \sum_{n=0}^q (-1)^{(s+1)n} F_{sn}^2(x) = (-1)^{sq} F_s(x) \sum_{n=0}^q (-1)^{sn} F_{2sn}(x) = F_{s(q+1)}(x) F_{sq}(x). \quad (94)$$

Observe also that, from (89) and (93), we can write

$$\begin{aligned} \frac{1}{F_{3s}^2(x)} \sum_{n=0}^q (-1)^{(s+1)(n+q)} F_{3sn}^2(x) &= \frac{1}{F_{6s}(x)} \sum_{n=0}^q (-1)^{s(n+q)} F_{6sn}(x) \\ &= \left(\begin{smallmatrix} q+1 \\ 6 \end{smallmatrix} \right)_{F_s(x)} + \left(\begin{smallmatrix} q+5 \\ 6 \end{smallmatrix} \right)_{F_s(x)} \\ &\quad + \left((-1)^{s+1} L_{4s}(x) - L_{2s}(x) \right) \left(\left(\begin{smallmatrix} q+2 \\ 6 \end{smallmatrix} \right)_{F_s(x)} + \left(\begin{smallmatrix} q+4 \\ 6 \end{smallmatrix} \right)_{F_s(x)} \right) \\ &\quad + (-1)^s \left(L_{6s}(x) + L_s^2(x) \right) \left(\begin{smallmatrix} q+3 \\ 6 \end{smallmatrix} \right)_{F_s(x)}. \end{aligned} \quad (95)$$

Some numerical examples of identities (90), (92) and (95), corresponding to $x = 1$ and $s = 1$ or $s = 2$, are the following

$$\begin{aligned} \frac{1}{4} \sum_{n=0}^q F_{3n}^2 &= \frac{(-1)^q}{8} \sum_{n=0}^q (-1)^n F_{6n} \\ &= \left(\begin{smallmatrix} q+1 \\ 6 \end{smallmatrix} \right)_F + \left(\begin{smallmatrix} q+5 \\ 6 \end{smallmatrix} \right)_F + 4 \left(\left(\begin{smallmatrix} q+2 \\ 6 \end{smallmatrix} \right)_F + \left(\begin{smallmatrix} q+4 \\ 6 \end{smallmatrix} \right)_F \right) - 19 \left(\begin{smallmatrix} q+3 \\ 6 \end{smallmatrix} \right)_F. \end{aligned} \quad (96)$$

$$\sum_{n=0}^q (-1)^n F_{2n}^3 = (-1)^q \left(\left(\begin{smallmatrix} q+1 \\ 6 \end{smallmatrix} \right)_F + \left(\begin{smallmatrix} q+5 \\ 6 \end{smallmatrix} \right)_F + 13 \left(\left(\begin{smallmatrix} q+2 \\ 6 \end{smallmatrix} \right)_F + \left(\begin{smallmatrix} q+4 \\ 6 \end{smallmatrix} \right)_F \right) + 44 \left(\begin{smallmatrix} q+3 \\ 6 \end{smallmatrix} \right)_F \right). \quad (97)$$

$$\sum_{n=0}^q F_n^6 = \left(\begin{smallmatrix} q+1 \\ 6 \end{smallmatrix} \right)_F + \left(\begin{smallmatrix} q+5 \\ 6 \end{smallmatrix} \right)_F - 11 \left(\left(\begin{smallmatrix} q+2 \\ 6 \end{smallmatrix} \right)_F + \left(\begin{smallmatrix} q+4 \\ 6 \end{smallmatrix} \right)_F \right) - 64 \left(\begin{smallmatrix} q+3 \\ 6 \end{smallmatrix} \right)_F. \quad (98)$$

$$\sum_{n=0}^q (-1)^{n+q} F_{2n}^6 = \binom{q+1}{6}_{F_2} + \binom{q+5}{6}_{F_2} + 351 \left(\binom{q+2}{6}_{F_2} + \binom{q+4}{6}_{F_2} \right) + 5056 \binom{q+3}{6}_{F_2}. \quad (99)$$

We comment in passing that (98) is an example given in [7] (p. 259). In that work we wrote: “It seems that the sum $\sum_{n=0}^q F_n^k$ is equal to [the sum of] $k-1$ Fibonomials (times some constants) if and only if $k \equiv 2 \pmod{4}$. We leave this affirmation as a conjecture.” The conjecture is now proved: the sum $\sum_{n=0}^q F_n^k$ corresponds to our case $t = s = 1$ (and of course $x = 1$) of proposition 1, and thus to the case (b), where $k \equiv 2 \pmod{4}$.

The simplest example from the case (a) of proposition 3, corresponding to $k = 4$ and $t = 1$, is the following identity valid for any $s \in \mathbb{N}$

$$\sum_{n=0}^q (-1)^{n+q} F_{sn}^4(x) = F_s^4(x) \left(\binom{q+1}{4}_{F_s(x)} + \binom{q+3}{4}_{F_s(x)} + (3(-1)^s L_{2s}(x) + 4) \binom{q+2}{4}_{F_s(x)} \right). \quad (100)$$

With some patience one can see that the case $x = 1$ of (100) is the identity

$$\sum_{n=0}^q (-1)^n F_{sn}^4 = \frac{(-1)^q F_{sq} F_{s(q+1)} (L_s L_{sq} L_{s(q+1)} - 4L_{2s})}{5L_s L_{2s}},$$

demonstrated in the year 2000 by Melham [6].

An example from the case (d) of proposition 3, corresponding to $t = k = 2$, is the following identity valid for any $s \in \mathbb{N}$

$$\sum_{n=0}^q (-1)^{n+q} F_{2sn}^2(x) = F_{2s}^2(x) \left(\binom{q+1}{4}_{F_s(x)} + (-1)^{s+1} L_{2s}(x) \binom{q+2}{4}_{F_s(x)} + \binom{q+3}{4}_{F_s(x)} \right). \quad (101)$$

The case $s = x = 1$ of (101) can be written as

$$\sum_{n=0}^q (-1)^n F_{2n}^2 = \frac{1}{3} ((-1)^q L_{2q+1} + 1) F_{q+1} F_q. \quad (102)$$

From the case $s = x = 1$ of (100) and (101), we can see that

$$\sum_{n=0}^q (-1)^n (F_{2n}^2 - F_n^4) = \frac{4}{3} (-1)^q F_{q+2} F_{q+1} F_q F_{q-1}. \quad (103)$$

Proposition 5 *The alternating sum $\sum_{n=0}^q (-1)^n L_{tsn}^k(x)$ can be written as a linear combination of the s -Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m = 1, 2, \dots, tk$, according to (72), if and only if s and k are odd positive integers and $t \equiv 2 \pmod{4}$.*

Proof. We will show that in the case stated in the proposition we have $\tilde{\Omega}_1(x, 1) < \infty$ and $\Omega_2(x, 1) = 0$, which implies (74), and that in all the remaining cases (74) does not hold. Since s and k are odd, and $t \equiv 2 \pmod{4}$, the factor $\left(z + (-1)^{\frac{tsk}{2}}\right)$ of (80) is $(z - 1)$, so we have $\Omega_2(x, 1) = 0$. We just have to check that $\tilde{\Omega}_1(x, 1) < \infty$. If in (78) we set $z = 1$ and replace k by $2k - 1$, we get for $t \equiv 2 \pmod{4}$ and s odd that

$$\begin{aligned} \tilde{\Omega}_1(x, 1) &= \sum_{l=0}^{2k-1} \binom{2k-1}{l} \frac{1}{1 + \alpha^{lts}(x) \beta^{(2k-1-l)ts}(x)} \\ &= \sum_{l=0}^{k-1} \binom{2k-1}{l} \left(\frac{1}{1 + \alpha^{lts}(x) \beta^{(2k-1-l)ts}(x)} + \frac{1}{1 + \alpha^{(2k-1-l)ts}(x) \beta^{lts}(x)} \right) \\ &= \sum_{l=0}^{k-1} \binom{2k-1}{l} \\ &= 4^{k-1}. \end{aligned}$$

This ends the proof of the sufficiency of the given condition. We have the following list of the parity possibilities for t , k and s .

	t	k	s
(i)	e ₁	o	o
(ii)	e ₂	o	o
(iii)	e	o	e
(iv)	e	e	o

	t	k	s
(v)	e	e	e
(vi)	o	e	e
(vii)	o	e	o
(viii)	o	o	e
(ix)	o	o	o

We will show that in the cases (ii) to (ix), expression (74) does not hold. Leaving the details to the reader, we have

(*) For $t = 4, k = 1$ (cases (ii) and (iii)) we have for any s ,

$$\sum_{i=0}^4 \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i} \binom{5}{j}_{F_s(x)} L_{4s(i-j)}(x) = -2L_s^2(x) L_{2s}^2(x). \quad (104)$$

(*) For $t = 2, k = 2$ (cases (iv) and (v)) we have for any s ,

$$\sum_{i=0}^4 \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i} \binom{5}{j}_{F_s(x)} L_{2s(i-j)}^2(x) = -4L_s^2(x) L_{2s}^2(x). \quad (105)$$

(*) For $t = 1, k = 2$ (cases (vi) and (vii)) we have for any s ,

$$\sum_{i=0}^2 \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i} \binom{3}{j}_{F_s(x)} L_{s(i-j)}^2(x) = -(3(-1)^s + 1)(L_{2s}(x) + 2). \quad (106)$$

(*) For $t = 1, k = 1$ (cases (viii) and (ix)) we have for any s ,

$$\sum_{i=0}^1 \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i} \binom{2}{j}_{F_s(x)} L_{s(i-j)}(x) = (-1)^s (L_s(x) + 2). \quad (107)$$

This ends the proof. ■

Corollary 6 *If $t \equiv 2 \pmod{4}$ and k is odd, we have the following identity valid for any $s \in \mathbb{N}$*

$$\begin{aligned} & \sum_{n=0}^q (-1)^{sn} L_{tsn}^k(x) \\ &= (-1)^{s+1} \sum_{m=1}^{tk} \sum_{i=0}^{tk-m} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}+s(i+tk+q+m)} \binom{tk+1}{j}_{F_s(x)} L_{ts(i-j)}^k(x) \binom{q+m}{tk}_{F_s(x)}. \end{aligned} \quad (108)$$

Proof. When $t \equiv 2 \pmod{4}$ and k is odd, formula (108) with s even, gives the result (58) of case (a) of proposition 2 (which is valid for t even, k odd and s even). Similarly, if $t \equiv 2 \pmod{4}$ and k is odd, formula (108) with s odd, gives the result (72) of proposition 5 (which is valid for $t \equiv 2 \pmod{4}$, k odd and s odd). ■

Some examples of (108) are the following identities, valid for any $s \in \mathbb{N}$.

With $t = 2$ and $k = 1$, identity (108) looks as

$$\sum_{n=0}^q (-1)^{s(n+q)} L_{2sn}(x) = 2 \binom{q+2}{2}_{F_s(x)} - L_{2s}(x) \binom{q+1}{2}_{F_s(x)}, \quad (109)$$

which can be written as

$$\sum_{n=0}^q (-1)^{s(n+q)} L_{2sn}(x) = \frac{1}{F_s(x)} L_{sq}(x) F_{s(q+1)}(x). \quad (110)$$

Comparing (110) with (66) we see that

$$\begin{aligned} \sum_{n=0}^q L_{4sn}(x) &= \frac{1}{F_{2s}(x)} L_{2sq}(x) F_{2s(q+1)}(x) \\ &= 2 \binom{q+4}{4}_{F_s(x)} + \left(-L_{4s}(x) + 2(-1)^{s+1} L_{2s}(x) \right) \binom{q+3}{4}_{F_s(x)} \\ &\quad + (-1)^s (L_{6s}(x) + L_{2s}(x) + 2(-1)^s) \binom{q+2}{4}_{F_s(x)} - L_{4s}(x) \binom{q+1}{4}_{F_s(x)}. \end{aligned} \quad (111)$$

The case $s = x = 1$ of (111) is

$$\sum_{n=0}^q L_{4n} = L_{2q} F_{2(q+1)} = 2 \binom{q+4}{4}_F - \binom{q+3}{4}_F - 19 \binom{q+2}{4}_F - 7 \binom{q+1}{4}_F.$$

With $t = 2$ and $k = 3$, identity (108) looks as

$$\begin{aligned} &\sum_{n=0}^q (-1)^{s(n+q)} L_{2sn}^3(x) \\ &= -L_{2s}^3(x) \binom{q+1}{6}_{F_s(x)} + L_{4s}(x) ((-1)^s L_{6s}(x) + 4L_{4s}(x) + 3(-1)^s L_{2s}(x) + 4) \binom{q+2}{6}_{F_s(x)} \\ &\quad + 4(-1)^{s+1} L_{4s}(x) (L_{8s}(x) + L_{4s}(x) + 2(-1)^s L_{2s}(x) + 2) \binom{q+3}{6}_{F_s(x)} \\ &\quad + ((-1)^s L_{2s}(x) (7L_{8s}(x) + 4(-1)^s L_{6s}(x) - 2L_{4s}(x) + 14) + 16) \binom{q+4}{6}_{F_s(x)} \\ &\quad - (L_{2s}(x) (7L_{4s}(x) + 8(-1)^s L_{2s}(x) - 2) + 16(-1)^{s+1}) \binom{q+5}{6}_{F_s(x)} + 8 \binom{q+6}{6}_{F_s(x)}. \end{aligned} \quad (112)$$

With $t = 6$ and $k = 1$, identity (108) looks as

$$\begin{aligned} &\sum_{n=0}^q (-1)^{s(n+q)} L_{6sn}(x) \\ &= -L_{6s}(x) \binom{q+1}{6}_{F_s(x)} + L_{4s}(x) ((-1)^s L_{6s}(x) + L_{4s}(x) + 1) \binom{q+2}{6}_{F_s(x)} \\ &\quad + (-1)^{s+1} L_{4s}(x) (L_{8s}(x) + L_{4s}(x) + 2(-1)^s L_{2s}(x) + 2) \binom{q+3}{6}_{F_s(x)} \\ &\quad + (L_{2s}(x) ((-1)^s L_{8s}(x) + L_{2s}(x) (L_{4s}(x) + (-1)^s L_{2s}(x) - 1)) + 4) \binom{q+4}{6}_{F_s(x)} \\ &\quad - (L_{2s}(x) (L_{4s}(x) + 2(-1)^s L_{2s}(x) + 1) + 4(-1)^{s+1}) \binom{q+5}{6}_{F_s(x)} + 2 \binom{q+6}{6}_{F_s(x)}. \end{aligned} \quad (113)$$

Note that according to (110) we also have

$$\sum_{n=0}^q (-1)^{s(n+q)} L_{6sn}(x) = \frac{1}{F_{3s}(x)} L_{3sq}(x) F_{3s(q+1)}(x).$$

The following identities are numerical examples from (112) and (113):

$$(-1)^q \sum_{n=0}^q (-1)^n L_{2n}^3 = -27 \binom{q+1}{6}_F + 35 \binom{q+2}{6}_F + 1400 \binom{q+3}{6}_F - 755 \binom{q+4}{6}_F - 85 \binom{q+5}{6}_F + 8 \binom{q+6}{6}_F. \quad (114)$$

$$\sum_{n=0}^q L_{12n} = -322 \binom{q+1}{6}_{F_2} + 17390 \binom{q+2}{6}_{F_2} - 106690 \binom{q+3}{6}_{F_2} + 18050 \binom{q+4}{6}_{F_2} - 430 \binom{q+5}{6}_{F_2} + 2 \binom{q+6}{6}_{F_2}. \quad (115)$$

5 Sums of powers of Gibonacci polynomials

Corollary 4 shows intersections on results for sums of powers $\sum_{n=0}^q F_{tsn}^k(x)$ and alternating sums of powers $\sum_{n=0}^q (-1)^n F_{tsn}^k(x)$ (both results within the Fibonacci world), and similarly, corollary 6 shows intersections on results for sums of powers $\sum_{n=0}^q L_{tsn}^k(x)$ and alternating sums of powers $\sum_{n=0}^q (-1)^n L_{tsn}^k(x)$ (both results within the Lucas world). Surprisingly, there are also some non-empty intersections between the Fibonacci world and the Lucas world, that invite us to think about results for sums of powers of Gibonacci polynomial sequences. In this section we give two propositions as examples of this kind of general results. The strategy of the corresponding proofs will be the same used in sections 3 and 4.

By using (4) we can see that

$$G_{tsn+\eta}^k(x) = \frac{c_2^k(x)}{(x^2+4)^{\frac{k}{2}}} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \left(\frac{c_1(x)}{c_2(x)} \right)^l \left(\alpha^{tsl}(x) \beta^{ts(k-l)}(x) \right)^n, \quad (116)$$

where $c_1(x) = (G_1(x) \alpha(x) + G_0(x)) \alpha^{\eta-1}(x)$ and $c_2(x) = (G_1(x) \beta(x) + G_0(x)) \beta^{\eta-1}(x)$. Thus, according to (18), the Z transform of the sequence $G_{tsn+\eta}^k(x)$ is

$$\mathcal{Z}(G_{tsn+\eta}^k(x)) = \frac{c_2^k(x)}{(x^2+4)^{\frac{k}{2}}} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \left(\frac{c_1(x)}{c_2(x)} \right)^l \frac{z}{z - \alpha^{tsl}(x) \beta^{ts(k-l)}(x)}, \quad (117)$$

and the Z transform of the alternating sequence $(-1)^n G_{tsn+\eta}^k(x)$ is

$$\mathcal{Z}((-1)^n G_{tsn+\eta}^k(x)) = \frac{c_2^k(x)}{(x^2+4)^{\frac{k}{2}}} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \left(\frac{c_1(x)}{c_2(x)} \right)^l \frac{z}{z + \alpha^{tsl}(x) \beta^{ts(k-l)}(x)}. \quad (118)$$

In [10] we proved that (117) can be written as

$$\mathcal{Z}(G_{tsn+\eta}^k(x)) = z \frac{\sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} G_{ts(i-j)+\eta}^k(x) z^{tk-i}}{\sum_{i=0}^{tk+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{tk+1}{i}_{F_s(x)} z^{tk+1-i}}, \quad (119)$$

(which includes expressions (27) and (28) as particular cases), and then we have also the following expression for (118)

$$\mathcal{Z}((-1)^n G_{tsn+\eta}^k(x)) = z \frac{\sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i} \binom{tk+1}{j}_{F_s(x)} G_{ts(i-j)+\eta}^k(x) z^{tk-i}}{\sum_{i=0}^{tk+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}+i} \binom{tk+1}{i}_{F_s(x)} z^{tk+1-i}}, \quad (120)$$

By using (119), we proved in [10] that the sequence $G_{tsn+\eta}^k(x)$ can be written as a linear combination of the s -Fibopolynomials $\binom{n+tk-i}{tk}_{F_s(x)}$, $i = 0, 1, \dots, tk$, according to

$$G_{stn+\eta}^k(x) = (-1)^{s+1} \sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} G_{st(i-j)+\eta}^k(x) \binom{n+tk-i}{tk}_{F_s(x)}. \quad (121)$$

(Formulas (29) and (29) are particular cases of (121).) From (121) we see that

$$\sum_{n=0}^q G_{stn+\eta}^k(x) = (-1)^{s+1} \sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} G_{st(i-j)+\eta}^k(x) \sum_{n=0}^q \binom{n+tk-i}{tk}_{F_s(x)},$$

and

$$\sum_{n=0}^q (-1)^n G_{stn+\eta}^k(x) = (-1)^{s+1} \sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} G_{st(i-j)+\eta}^k(x) \sum_{n=0}^q (-1)^n \binom{n+tk-i}{tk}_{F_s(x)},$$

which can be written as

$$\begin{aligned} & \sum_{n=0}^q G_{tsn+\eta}^k(x) \\ &= (-1)^{s+1} \sum_{m=1}^{tk} \sum_{i=0}^{tk-m} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} G_{ts(i-j)+\eta}^k(x) \binom{q+m}{tk}_{F_s(x)} \\ & \quad + (-1)^{s+1} \sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} G_{ts(i-j)+\eta}^k(x) \sum_{n=0}^q \binom{n}{tk}_{F_s(x)}, \end{aligned} \quad (122)$$

and

$$\begin{aligned} & \sum_{n=0}^q (-1)^n G_{tsn+\eta}^k(x) \\ &= (-1)^{s+1} \sum_{m=1}^{tk} \sum_{i=0}^{tk-m} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2} + i + tk + q + m} \binom{tk+1}{j}_{F_s(x)} G_{ts(i-j)+\eta}^k(x) \binom{q+m}{tk}_{F_s(x)} \\ & \quad + (-1)^{s+1} \sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2} + i + tk} \binom{tk+1}{j}_{F_s(x)} G_{ts(i-j)+\eta}^k(x) \sum_{n=0}^q \binom{n}{tk}_{F_s(x)}, \end{aligned} \quad (123)$$

respectively. (Formulas (41), (57), (69) and (70) are particular cases of (122) and (123).) Thus, from (122) we see at once that the sum $\sum_{n=0}^q G_{tsn+\eta}^k(x)$ can be written as a linear combination of the s -Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m = 1, 2, \dots, tk$, according to

$$\sum_{n=0}^q G_{tsn+\eta}^k(x) = (-1)^{s+1} \sum_{m=1}^{tk} \sum_{i=0}^{tk-m} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} G_{ts(i-j)+\eta}^k(x) \binom{q+m}{tk}_{F_s(x)}, \quad (124)$$

if and only if

$$\sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} G_{ts(i-j)+\eta}^k(x) = 0. \quad (125)$$

Similarly, from (123) we see that the alternating sum $\sum_{n=0}^q (-1)^n G_{tsn+\eta}^k(x)$ can be written as a linear combination of the s -Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m = 1, 2, \dots, tk$, according to

$$\begin{aligned} & \sum_{n=0}^q (-1)^n G_{tsn+\eta}^k(x) \\ = & (-1)^{s+1} \sum_{m=1}^{tk} \sum_{i=0}^{tk-m} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2} + i + tk + q + m} \binom{tk+1}{j}_{F_s(x)} G_{ts(i-j)+\eta}^k(x) \binom{q+m}{tk}_{F_s(x)}, \end{aligned} \quad (126)$$

if and only if

$$\sum_{i=0}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2} + i} \binom{tk+1}{j}_{F_s(x)} G_{ts(i-j)+\eta}^k(x) = 0. \quad (127)$$

Our first result was inspired by the intersection between case (c) of proposition 1 and case (b) of proposition 2.

Proposition 7 *If $t \equiv 0 \pmod{4}$, k is odd, and s is any natural number, we have the following identity valid for any $\eta \in \mathbb{Z}$*

$$\begin{aligned} & \sum_{n=0}^q G_{tsn+\eta}^k(x) \\ = & (-1)^{s+1} \sum_{m=1}^{tk} \sum_{i=0}^{tk-m} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} G_{ts(i-j)+\eta}^k(x) \binom{q+m}{tk}_{F_s(x)}. \end{aligned} \quad (128)$$

Proof. First of all observe that according to (117) and (119), the conditions

$$\sum_{i=0}^{tk+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{tk+1}{i}_{F_s(x)} = 0, \quad (129)$$

and

$$\sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \left(\frac{c_1(x)}{c_2(x)} \right)^l \frac{1}{1 - \alpha^{tsl}(x) \beta^{ts(k-l)}(x)} < \infty, \quad (130)$$

imply (125). Observe also that if $\eta = 0$ and $G = F$, formula (128) is just the case (c) of proposition 1, and if $\eta = 0$ and $G = L$, formula (128) is just the case (b) of proposition 2. In the corresponding proofs of these cases we proved that (129) holds (observe that (129) does not involve the Gibonacci sequence). Thus, it remains to prove that (130) holds. By writing k as $2k-1$, and using that $t \equiv 0 \pmod{4}$, we have that

$$\begin{aligned} & \sum_{l=0}^{2k-1} \binom{2k-1}{l} (-1)^l \left(\frac{c_1(x)}{c_2(x)} \right)^l \frac{1}{1 - \alpha^{tsl}(x) \beta^{ts(2k-1-l)}(x)} \\ = & \sum_{l=0}^{k-1} \binom{2k-1}{l} \frac{(-1)^{l+1}}{2 - L_{ts(2k-1-2l)}(x)} \\ & \times \left(\left(\frac{c_1(x)}{c_2(x)} \right)^{2k-1-l} \left(1 - \alpha^{tsl}(x) \beta^{ts(2k-1-l)}(x) \right) - \left(\frac{c_1(x)}{c_2(x)} \right)^l \left(1 - \alpha^{ts(2k-1-l)}(x) \beta^{tsl}(x) \right) \right). \end{aligned} \quad (131)$$

Now it is clear that (130) holds. ■

For example, if $t = 4$ and $k = 1$, we have the following identity valid for any $\eta \in \mathbb{Z}$ and any $s \in \mathbb{N}$

$$\begin{aligned} \sum_{n=0}^q G_{4sn+\eta}(x) &= - \left(\begin{array}{c} G_{16s+\eta}(x) - \binom{5}{1}_{F_s(x)} (G_{12s+\eta}(x) - G_\eta(x)) \\ + (-1)^s \binom{5}{2}_{F_s(x)} (G_{8s+\eta}(x) - G_{4s+\eta}(x)) \end{array} \right) \binom{q+1}{4}_{F_s(x)} \\ &\quad + \left(\begin{array}{c} G_\eta(x) + G_{4s+\eta}(x) + G_{8s+\eta}(x) - \binom{5}{1}_{F_s(x)} (G_{4s+\eta}(x) + G_\eta(x)) \\ + (-1)^s \binom{5}{2}_{F_s(x)} G_\eta(x) \end{array} \right) \binom{q+2}{4}_{F_s(x)} \\ &\quad + \left(\left(1 - \binom{5}{1}_{F_s(x)} \right) G_\eta(x) + G_{4s+\eta}(x) \right) \binom{q+3}{4}_{F_s(x)} + G_\eta(x) \binom{q+4}{4}_{F_s(x)}, \end{aligned} \quad (132)$$

which includes as particular cases to (55) and (66). More particularly, if we consider the Gibonacci polynomial sequence $G_n(x)$, with $G_0(x) = x^2 + 1$ and $G_1(x) = x + 1$, identity (132) with $\eta = 3$ and $s = 1$ looks as

$$\begin{aligned} \sum_{n=0}^q G_{4n+3}(x) &= (x^3 - 1) \binom{q+1}{4}_{F(x)} + (x^5 + 4x^3 + 2x - 1) \binom{q+2}{4}_{F(x)} \\ &\quad + (2x^5 + x^4 + 7x^3 + 3x^2 + 4x + 1) \binom{q+3}{4}_{F(x)} \\ &\quad + (x^2 + 1)(2x + 1) \binom{q+4}{4}_{F(x)}, \end{aligned} \quad (133)$$

and with $s = 2$, we have

$$\begin{aligned} \sum_{n=0}^q G_{8n+3}(x) &= (x^7 + 4x^5 - x^4 + 4x^3 - 3x^2 + 2x - 1) \binom{q+1}{4}_{F_2(x)} \\ &\quad + \left(\begin{array}{c} -x^{11} - 8x^9 + x^8 - 22x^7 + 7x^6 - 26x^5 \\ + 15x^4 - 14x^3 + 11x^2 - 2x + 3 \end{array} \right) \binom{q+2}{4}_{F_2(x)} \\ &\quad - (x^7 + x^6 + 6x^5 + 6x^4 + 8x^3 + 9x^2 + 2x + 3) \binom{q+3}{4}_{F_2(x)} \\ &\quad + (x^2 + 1)(2x + 1) \binom{q+4}{4}_{F_2(x)}. \end{aligned} \quad (134)$$

Formulas (133) and (134) contain in turn the numerical identities (with $x = 1$)

$$\sum_{n=0}^q F_{4(n+1)} = 3 \left(\binom{q+2}{4}_F + 3 \binom{q+3}{4}_F + \binom{q+4}{4}_F \right), \quad (135)$$

(which is essentially (55)), and

$$\sum_{n=0}^q F_{4(2n+1)} = 3 \left(\binom{q+1}{4}_{F_2} + \binom{q+4}{4}_{F_2} - 6 \left(\binom{q+2}{4}_{F_2} + \binom{q+3}{4}_{F_2} \right) \right), \quad (136)$$

respectively.

The following result was inspired by the intersection of case (b) of corollary 4 (which in turn involves to case (a) of proposition 1 and to case (c) of proposition 3), and corollary 6 (which in turn involves to case (a) of proposition 2 and to proposition 5).

Proposition 8 *If $t \equiv 2 \pmod{4}$ and k is odd, we have the following identity valid for any $\eta \in \mathbb{Z}$ and $s \in \mathbb{N}$*

$$\begin{aligned} & \sum_{n=0}^q (-1)^{s(n+q)} G_{tsn+\eta}^k(x) \\ &= (-1)^{s+1} \sum_{m=1}^{tk} \sum_{i=0}^{tk-m} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2} + s(i+q+m)} \binom{tk+1}{j}_{F_s(x)} G_{ts(i-j)+\eta}^k(x) \binom{q+m}{tk}_{F_s(x)}. \end{aligned} \quad (137)$$

Proof. First observe that with $t \equiv 2 \pmod{4}$ and k odd we have, for s even

$$\sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \left(\frac{c_1(x)}{c_2(x)} \right)^l \frac{1}{1 - \alpha^{tsl}(x) \beta^{ts(k-l)}(x)} < \infty, \quad (138)$$

and, for s odd

$$\sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \left(\frac{c_1(x)}{c_2(x)} \right)^l \frac{1}{1 + \alpha^{tsl}(x) \beta^{ts(k-l)}(x)} < \infty. \quad (139)$$

To prove these facts are easy exercises left to the reader. Let us consider first the case of s even. We need to prove that if $t \equiv 2 \pmod{4}$ and k is odd, then we have (125). In this case the factorization of the denominator of (119) contains the factor $\left(z - (-1)^{\frac{kts}{2}} \right) = (z - 1)$, which is 0 for $z = 1$. This fact, together with (138) (and of course, together with (117) and (119)), give us the desired conclusion. On the other hand, if s is odd, we need to prove that if $t \equiv 2 \pmod{4}$ and k is odd, then we have (127). In this case the factorization of the denominator of (120) contains the factor $\left(z + (-1)^{\frac{kts}{2}} \right) = (z - 1)$, which is 0 for $z = 1$. This fact, together with (139) (and of course, together with (118) and (120)), give us the desired conclusion for this case. ■

For example, if $t = 2$ and $k = 1$, we obtain from (137) the following identity, valid for any $\eta \in \mathbb{Z}$ and $s \in \mathbb{N}$

$$\sum_{n=0}^q (-1)^{s(n+q)} G_{2sn+\eta}(x) = \left(\left((-1)^s - \binom{3}{1}_{F_s(x)} \right) G_{\eta}(x) + G_{2s+\eta}(x) \right) \binom{q+1}{2}_{F_s(x)} + G_{\eta}(x) \binom{q+2}{2}_{F_s(x)}. \quad (140)$$

(This identity includes as particular cases to (91) and (109).) More particularly, if $G_0(x) = x + 1$ and $G_1(x) = x^2 + x + 1$, we have from (140) with $\eta = 0$ and $s = 1$, the identity

$$\sum_{n=0}^q (-1)^{n+q} G_{2n}(x) = - \binom{q+1}{2}_{F(x)} + (x+1) \binom{q+2}{2}_{F(x)}, \quad (141)$$

and with $s = 2$, we get

$$\sum_{n=0}^q G_{4n}(x) = (-x^2 + x - 1) \binom{q+1}{2}_{F_2(x)} + (x+1) \binom{q+2}{2}_{F_2(x)}. \quad (142)$$

In particular (with $x = 1$) we have the following Fibonacci identities

$$\sum_{n=0}^q (-1)^{n+q} F_{2n+3} = - \binom{q+1}{2}_F + 2 \binom{q+2}{2}_F = (2F_{q+2} - F_q) F_{q+1}. \quad (143)$$

$$\sum_{n=0}^q F_{4n+3} = - \binom{q+1}{2}_{F_2} + 2 \binom{q+2}{2}_{F_2} = \frac{1}{3} (2F_{2(q+2)} - F_{2q}) F_{2(q+1)}. \quad (144)$$

In the case $t = 2$ and $k = 3$, identity (137) looks as

$$\begin{aligned}
& \sum_{n=0}^q (-1)^{s(n+q)} G_{2sn+\eta}^3(x) \\
&= \left(\begin{array}{l} (-1)^{s+1} G_{12s+\eta}^3(x) + \binom{7}{1}_{F_s(x)} ((-1)^s G_{10s+\eta}^3(x) - G_{\eta}^3(x)) \\ - \binom{7}{2}_{F_s(x)} (G_{8s+\eta}^3(x) + (-1)^{s+1} G_{2s+\eta}^3(x)) \\ + \binom{7}{3}_{F_s(x)} (G_{6s+\eta}^3(x) + (-1)^{s+1} G_{4s+\eta}^3(x)) \end{array} \right) \binom{q+1}{6}_{F_s(x)} \\
&+ \left(\begin{array}{l} (-1)^{s+1} G_{10s+\eta}^3(x) - G_{12s+\eta}^3(x) \\ + \binom{7}{1}_{F_s(x)} ((-1)^s G_{8s+\eta}^3(x) + G_{10s+\eta}^3(x) + (-1)^{s+1} G_{\eta}^3(x)) \\ - \binom{7}{2}_{F_s(x)} (G_{6s+\eta}^3(x) + (-1)^{s+1} G_{\eta}^3(x) + (-1)^s G_{8s+\eta}^3(x) - G_{2s+\eta}^3(x)) \\ + (-1)^{s+1} \binom{7}{3}_{F_s(x)} (G_{2s+\eta}^3(x) - G_{6s+\eta}^3(x)) \end{array} \right) \binom{q+2}{6}_{F_s(x)} \\
&+ \left(\begin{array}{l} (-1)^s G_{\eta}^3(x) + G_{2s+\eta}^3(x) + (-1)^s G_{4s+\eta}^3(x) + G_{6s+\eta}^3(x) \\ - \binom{7}{1}_{F_s(x)} (G_{\eta}^3(x) + G_{4s+\eta}^3(x) + (-1)^s G_{2s+\eta}^3(x)) \\ + \binom{7}{2}_{F_s(x)} (G_{\eta}^3(x) + (-1)^s G_{2s+\eta}^3(x)) + (-1)^{s+1} \binom{7}{3}_{F_s(x)} G_{\eta}^3(x) \end{array} \right) \binom{q+3}{6}_{F_s(x)} \\
&+ \left(\begin{array}{l} G_{\eta}^3(x) + (-1)^s G_{2s+\eta}^3(x) + G_{4s+\eta}^3(x) \\ + \binom{7}{1}_{F_s(x)} ((-1)^{s+1} G_{\eta}^3(x) - G_{2s+\eta}^3(x)) + (-1)^s \binom{7}{2}_{F_s(x)} G_{\eta}^3(x) \end{array} \right) \binom{q+4}{6}_{F_s(x)} \\
&+ \left(\begin{array}{l} (-1)^s - \binom{7}{1}_{F_s(x)} \\ \left((-1)^s - \binom{7}{1}_{F_s(x)} \right) G_{\eta}^3(x) + G_{2s+\eta}^3(x) \end{array} \right) \binom{q+5}{6}_{F_s(x)} \\
&+ G_{\eta}^3(x) \binom{q+6}{6}_{F_s(x)}.
\end{aligned} \tag{145}$$

which includes as particular cases identities (92) and (112). For example, if $G_0(x) = x$, $G_1(x) = x + 1$, $s = 1$ and $\eta = 2$, we have from (145) the following identity

$$\sum_{n=0}^q (-1)^{n+q} G_{2n+2}^3(x) = x^3 \left(\begin{array}{l} - \binom{q+1}{6}_{F(x)} + \binom{3x^5 + 2x^4 + x^3}{+6x^2 + 2} \binom{q+2}{6}_{F(x)} \\ + \binom{3x^9 + 3x^8 + 13x^7 + 19x^6}{+38x^4 - 15x^3 + 22x^2 + 6x + 2} \binom{q+3}{6}_{F(x)} \\ - \binom{3x^{10} + 10x^9 + 24x^8 + 57x^7}{+68x^6 + 79x^5 + 82x^4 + 8x^3} \binom{q+4}{6}_{F(x)} \\ + \binom{x^7 + 3x^6 + 6x^5 + 20x^4}{+33x^3 + 30x^2 + 30x + 11} \binom{q+5}{6}_{F(x)} \\ + (x+2)^3 \binom{q+6}{6}_{F(x)} \end{array} \right).$$

In particular (with $x = 1$) we have

$$\sum_{n=0}^q (-1)^{n+q} F_{2n+4}^3 = - \binom{q+1}{6}_F + 14 \binom{q+2}{6}_F + 91 \binom{q+3}{6}_F - 337 \binom{q+4}{6}_F + 134 \binom{q+5}{6}_F + 27 \binom{q+6}{6}_F.$$

Another numerical example from (145) is the following identity, valid for $a, b \in \mathbb{R}$

$$\begin{aligned} & \sum_{n=0}^q (aF_{4n+3} + bL_{4n+3})^3 \\ = & (b-a)^3 \binom{q+1}{6}_{F_2} + (251a^3 - 303a^2b - 687ab^2 + 955b^3) \binom{q+2}{6}_{F_2} \\ & - 8(1253a^3 - 816a^2b - 5016ab^2 + 5470b^3) \binom{q+3}{6}_{F_2} \\ & + (20657a^3 + 51069a^2b - 57837ab^2 - 155585b^3) \binom{q+4}{6}_{F_2} \\ & + (-811a^3 - 3345a^2b - 3297ab^2 + 325b^3) \binom{q+5}{6}_{F_2} + 8(a+2b)^3 \binom{q+6}{6}_{F_2}. \end{aligned}$$

6 Further comments

Expression (41) can be written as

$$\begin{aligned} & \sum_{n=0}^q \left(F_{tsn}^k(x) + (-1)^s \sum_{i=1}^{tk} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x) \binom{n}{tk}_{F_s(x)} \right) \\ = & (-1)^{s+1} \sum_{m=1}^{tk-1} \sum_{i=1}^{tk-m} \sum_{j=0}^i (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x) \binom{q+m}{tk}_{F_s(x)}. \end{aligned} \quad (146)$$

Similar formulas to (146) can be obtained by rewriting expressions (57), (69) and (70). These formulas give us the sums of powers of Fibonacci or Lucas polynomials in terms of s -Fibopolynomials, if and only if the double summatory of the parenthesis of the left-hand side is equal to zero: these are precisely the conditions (43), (59), (73) and (74). However, when these conditions are not satisfied we can even have new interesting identities (or rediscoveries of identities of sections 3 and 4). This is the first part of what we want to do in this section. The second part is about some examples of identities obtained as derivatives of some of our previous results.

Let us begin with (146). If we set $t = 2$ and $k = 1$ in (146), we obtain

$$\sum_{n=0}^q \left(F_{2sn}(x) - F_{2s}(x) \left(1 + (-1)^{s+1} \right) \binom{n}{2}_{F_s(x)} \right) = F_{2s}(x) \binom{q+1}{2}_{F_s(x)}, \quad (147)$$

In particular, if s is even, we have (91), but if s is odd what we have is

$$\sum_{n=0}^q (F_s(x) L_{sn}(x) - 2F_{s(n-1)}(x)) F_{sn}(x) = F_{s(q+1)}(x) F_{sq}(x). \quad (148)$$

If we set $t = 1$ and $k = 2$ in (146), we obtain

$$\sum_{n=0}^q \left(F_{sn}^2(x) + F_s^2(x) \left((-1)^{s+1} - 1 \right) \binom{n}{2}_{F_s(x)} \right) = F_s^2(x) \binom{q+1}{2}_{F_s(x)}. \quad (149)$$

In particular, if s is odd we have (88), but if s is even what we have is

$$\sum_{n=0}^q (L_s(x) F_{sn}(x) - 2F_{s(n-1)}(x)) F_{sn}(x) = F_{s(q+1)}(x) F_{sq}(x). \quad (150)$$

If we set $t = 1$ and $k = 3$ in (146), we obtain the identity

$$\begin{aligned} & \sum_{n=0}^q \left(F_{sn}^3(x) + \frac{(-1)^{s+1} (2L_s(x) + 1 + (-1)^s)}{L_s(x) (L_s^2(x) + (-1)^{s+1})} F_{sn}(x) F_{s(n-1)}(x) F_{s(n-2)}(x) \right) \\ &= F_s^3(x) \left((2(-1)^s L_s(x) + 1) \binom{q+1}{3}_{F_s(x)} + \binom{q+2}{3}_{F_s(x)} \right). \end{aligned} \quad (151)$$

Now let us consider some examples taken from (57). If we set $t = 2$ and $k = 1$ we get

$$\begin{aligned} & \sum_{n=0}^q \left(L_{2sn}(x) + (1 + (-1)^{s+1}) L_s^2(x) \binom{n}{2}_{F_s(x)} \right) \\ &= \left(2(1 + (-1)^{s+1}) - L_{2s}(x) \right) \binom{q+1}{2}_{F_s(x)} + 2 \binom{q+2}{2}_{F_s(x)}. \end{aligned} \quad (152)$$

In particular, if s is even we have (109), but if s is odd what we have is

$$\sum_{n=0}^q \left(L_{2sn}(x) + 2L_s^2(x) \binom{n}{2}_{F_s(x)} \right) = (4 - L_{2s}(x)) \binom{q+1}{2}_{F_s(x)} + 2 \binom{q+2}{2}_{F_s(x)}. \quad (153)$$

If we set $t = 1$ and $k = 2$ in (57), we obtain

$$\begin{aligned} & \sum_{n=0}^q \left(L_{sn}^2(x) - (6 + 2(-1)^{s+1} - ((-1)^{s+1} + 3) L_{2s}(x)) \binom{n}{2}_{F_s(x)} \right) \\ &= \left(2(-1)^{s+1} + 4 - 3L_{2s}(x) \right) \binom{q+1}{2}_{F_s(x)} + 4 \binom{q+2}{2}_{F_s(x)}. \end{aligned} \quad (154)$$

The case $s = x = 1$ of (154) can be written as

$$\sum_{n=0}^q (L_n^2 + 4F_n F_{n-1}) = (4F_{q+2} - 3F_q) F_{q+1}. \quad (155)$$

Now we consider some examples from (69). The simplest case is when $t = 1$ and $k = 2$. We get

$$\sum_{n=0}^q (-1)^n \left(F_{sn}^2(x) + \frac{1 + (-1)^{s+1}}{L_s(x)} F_{sn}(x) F_{s(n-1)}(x) \right) = \frac{(-1)^q}{L_s(x)} F_{sq}(x) F_{s(q+1)}(x). \quad (156)$$

The case s even of (156) is contained in (94). If s is odd what we get is

$$\sum_{n=0}^q (-1)^n (L_s(x) F_{sn}(x) + 2F_{s(n-1)}(x)) F_{sn}(x) = (-1)^q F_{sq}(x) F_{s(q+1)}(x), \quad (157)$$

or (by using that for s odd we have that $L_s(x) F_{sn}(x) + 2F_{s(n-1)}(x) = F_s(x) L_{sn}(x)$)

$$F_s(x) \sum_{n=0}^q (-1)^n F_{2sn}(x) = (-1)^q F_{sq}(x) F_{s(q+1)}(x),$$

which is also contained in (94).

If we set $t = 1$ and $k = 3$ in (69) we get

$$\begin{aligned} & \sum_{n=0}^q (-1)^{n+q} \left(F_{sn}^3(x) - \frac{2(-1)^{s+1} L_s(x) + (-1)^s + 1}{L_s(x) (L_s^2(x) + (-1)^{s+1})} F_{sn}(x) F_{s(n-1)}(x) F_{s(n-2)}(x) \right) \\ &= F_s^3(x) \left((2(-1)^s L_s(x) - 1) \binom{q+1}{3}_{F_s(x)} + \binom{q+2}{3}_{F_s(x)} \right). \end{aligned} \quad (158)$$

Comparing with (151) we see that for $r = 1, 2$ we have the identity

$$\begin{aligned} & \sum_{n=0}^q (-1)^{r(n+q)} \left(F_{sn}^3(x) + \frac{2(-1)^{s+r+1} L_s(x) + (-1)^{s+1} - 1}{L_s(x) (L_s^2(x) + (-1)^{s+1})} F_{sn}(x) F_{s(n-1)}(x) F_{s(n-2)}(x) \right) \\ &= F_s^3(x) \left((2(-1)^s L_s(x) + (-1)^r) \binom{q+1}{3}_{F_s(x)} + \binom{q+2}{3}_{F_s(x)} \right). \end{aligned} \quad (159)$$

Some particular cases of (159) when $x = 1$ are shown in the following table

	$r = 1$	$r = 2$
$s = 1$	$\sum_{n=0}^q (-1)^n (F_{2n-3} + F_{n-1} F_{n-2}) F_n$ $= (-1)^q \left(\binom{q+2}{3}_F - 3 \binom{q+1}{3}_F \right)$	$\sum_{n=0}^q (F_n^2 + F_{n-1} F_{n-2}) F_n$ $= \binom{q+2}{3}_F - \binom{q+1}{3}_F$
$s = 2$	$\sum_{n=0}^q (-1)^n (F_{2n}^2 + \frac{1}{6} F_{2(n-1)} F_{2(n-2)}) F_{2n}$ $= (-1)^q \left(\binom{q+2}{3}_{F_2} + 5 \binom{q+1}{3}_{F_2} \right)$	$\sum_{n=0}^q (F_{2n}^2 - \frac{1}{3} F_{2(n-1)} F_{2(n-2)}) F_{2n}$ $= \binom{q+2}{3}_{F_2} + 7 \binom{q+1}{3}_{F_2}$

Let us consider now some examples from identity (70). The case $t = 2$ and $k = 1$ is

$$\begin{aligned} & \sum_{n=0}^q (-1)^{n+q} \left(L_{2sn}(x) + ((-1)^{s+1} - 1) (L_{2s}(x) + 2) \binom{n}{2}_{F_s(x)} \right) \\ &= 2 \binom{q+2}{2}_{F_s(x)} + (-L_{2s}(x) + 2((-1)^{s+1} - 1)) \binom{q+1}{2}_{F_s(x)}. \end{aligned} \quad (160)$$

Comparing with (152), we conclude that for $r = 1, 2$ we have

$$\begin{aligned} & \sum_{n=0}^q (-1)^{r(n+q)} \left(L_{2sn}(x) + ((-1)^r + (-1)^{s+1}) (L_{2s}(x) + 2(-1)^{s(r+1)}) \binom{n}{2}_{F_s(x)} \right) \\ &= \left(2((-1)^r + (-1)^{s+1}) - L_{2s}(x) \right) \binom{q+1}{2}_{F_s(x)} + 2 \binom{q+2}{2}_{F_s(x)}. \end{aligned} \quad (161)$$

Observe that the case $r = s$ of (161) is (109). Two particular cases from (161) with $x = 1$ are the following:

For $r = 1$ and $s = 2$ we have

$$\sum_{n=0}^q (-1)^n (L_{4n} - 6F_{2n}F_{2(n-1)}) = \frac{(-1)^q}{3} (2F_{2(q+2)} - 11F_{2q}) F_{2(q+1)}. \quad (162)$$

For $r = 2$ and $s = 1$ we have

$$\sum_{n=0}^q (L_{2n} + 2F_nF_{n-1}) = (F_q + 2F_{q+2}) F_{q+1}. \quad (163)$$

If we set $t = 1$ and $k = 2$ in (70) we get

$$\begin{aligned} & \sum_{n=0}^q (-1)^n \left(L_{sn}^2(x) - (3 + (-1)^s) (L_{2s}(x) + 2) \binom{n}{2}_{F_s(x)} \right) \\ &= (-1)^{q+1} (3L_{2s}(x) + 2(2 + (-1)^s)) \binom{q+1}{2}_{F_s(x)} + 4(-1)^q \binom{q+2}{2}_{F_s(x)}. \end{aligned} \quad (164)$$

The case $s = x = 1$ of (164) can be written as

$$\sum_{n=0}^q (-1)^n L_{2n-3} = (-1)^q L_{q-3} F_{q+1}. \quad (165)$$

To end this work we want to present (in two propositions) some examples of identities obtained as derivatives of some of our previous results. We will use formulas (38) and (39).

Proposition 9 *The following identities hold*

$$\begin{aligned} & (-1)^{(s+1)q} 2L_s(x) \sum_{n=0}^q (-1)^{(s+1)n} nF_{2sn}(x) \\ &= 2qF_{s(2q+1)}(x) + F_{sq}(x) L_{s(q+1)}(x) - \frac{F_s(x)}{L_s(x)} (x^2 + 4) F_{s(q+1)}(x) F_{sq}(x). \end{aligned} \quad (166)$$

$$\begin{aligned} & (-1)^{sq} 2F_s(x) \sum_{n=0}^q (-1)^{sn} nL_{2sn}(x) \\ &= 2qF_{s(2q+1)}(x) + F_{sq}(x) L_{s(q+1)}(x) - \frac{L_s(x)}{F_s(x)} F_{s(q+1)}(x) F_{sq}(x). \end{aligned} \quad (167)$$

Proof. We will use (94), which contains two identities, namely

$$(-1)^{(s+1)q} L_s(x) \sum_{n=0}^q (-1)^{(s+1)n} F_{sn}^2(x) = F_{s(q+1)}(x) F_{sq}(x), \quad (168)$$

and

$$(-1)^{sq} F_s(x) \sum_{n=0}^q (-1)^{sn} F_{2sn}(x) = F_{s(q+1)}(x) F_{sq}(x). \quad (169)$$

First observe that

$$\begin{aligned}
& \frac{d}{dx} (F_{s(q+1)}(x) F_{sq}(x)) \\
&= \frac{1}{x^2+4} (F_{s(q+1)}(x) (sqL_{sq}(x) - xF_{sq}(x)) + F_{sq}(x) (s(q+1)L_{s(q+1)}(x) - xF_{s(q+1)}(x))) \\
&= \frac{1}{x^2+4} (2sqF_{s(2q+1)}(x) + sF_{sq}(x)L_{s(q+1)}(x) - 2xF_{s(q+1)}(x)F_{sq}(x)),
\end{aligned}$$

where we used that $L_{sq}(x)F_{s(q+1)}(x) + F_{sq}(x)L_{s(q+1)}(x) = 2F_{s(2q+1)}(x)$.

The derivative of the left-hand side of (168) is

$$\begin{aligned}
& (-1)^{(s+1)q} \left(L_s(x) \sum_{n=0}^q (-1)^{(s+1)n} 2F_{sn}(x) \frac{snL_{sn}(x) - xF_{sn}(x)}{x^2+4} + sF_s(x) \sum_{n=0}^q (-1)^{(s+1)n} F_{sn}^2(x) \right) \\
&= (-1)^{(s+1)q} \left(L_s(x) \sum_{n=0}^q \frac{(-1)^{(s+1)n}}{x^2+4} 2snF_{2sn}(x) + \left(sF_s(x) - \frac{2xL_s(x)}{x^2+4} \right) \sum_{n=0}^q (-1)^{(s+1)n} F_{sn}^2(x) \right) \\
&= (-1)^{(s+1)q} L_s(x) \sum_{n=0}^q \frac{(-1)^{(s+1)n}}{x^2+4} 2snF_{2sn}(x) + \left(sF_s(x) - \frac{2xL_s(x)}{x^2+4} \right) \frac{1}{L_s(x)} F_{s(q+1)}(x) F_{sq}(x).
\end{aligned}$$

Then, the derivative of (168) is

$$\begin{aligned}
& (-1)^{(s+1)q} L_s(x) \sum_{n=0}^q \frac{(-1)^{(s+1)n}}{x^2+4} 2snF_{2sn}(x) + \left(sF_s(x) - \frac{2xL_s(x)}{x^2+4} \right) \frac{1}{L_s(x)} F_{s(q+1)}(x) F_{sq}(x) \\
&= \frac{1}{x^2+4} (2sqF_{s(2q+1)}(x) + sF_{sq}(x)L_{s(q+1)}(x) - 2xF_{s(q+1)}(x)F_{sq}(x)),
\end{aligned}$$

from where (166) follows.

The derivative of the left-hand side of (169) is

$$\begin{aligned}
& \frac{(-1)^{sq}}{x^2+4} \left(F_s(x) \sum_{n=0}^q (-1)^{sn} (2snL_{2sn} - xF_{2sn}(x)) + (sL_s - xF_s(x)) \sum_{n=0}^q (-1)^{sn} F_{2sn}(x) \right) \\
&= \frac{(-1)^{sq}}{x^2+4} \left(F_s(x) \sum_{n=0}^q (-1)^{sn} 2snL_{2sn} + (sL_s - 2xF_s(x)) \sum_{n=0}^q (-1)^{sn} F_{2sn}(x) \right) \\
&= \frac{(-1)^{sq}}{x^2+4} F_s(x) \sum_{n=0}^q (-1)^{sn} 2snL_{2sn} + \frac{sL_s - 2xF_s(x)}{F_s(x)(x^2+4)} F_{s(q+1)}(x) F_{sq}(x).
\end{aligned}$$

Thus, the derivative of (169) is

$$\begin{aligned}
& \frac{(-1)^{sq}}{x^2+4} F_s(x) \sum_{n=0}^q (-1)^{sn} 2snL_{2sn} + \frac{sL_s - 2xF_s(x)}{F_s(x)(x^2+4)} F_{s(q+1)}(x) F_{sq}(x) \\
&= \frac{1}{x^2+4} (2sqF_{s(2q+1)}(x) + sF_{sq}(x)L_{s(q+1)}(x) - 2xF_{s(q+1)}(x)F_{sq}(x)),
\end{aligned}$$

from where (167) follows. ■

Proposition 10 *The following identity holds*

$$(-1)^{sq} F_s(x) \sum_{n=0}^q (-1)^{sn} nF_{2sn}(x) = \frac{1}{x^2+4} \left(\frac{(-1)^{s+1}}{F_s(x)} F_{2sq}(x) + qL_{s(2q+1)}(x) \right). \quad (170)$$

Proof. Identity (170) is the derivative of (110), together with

$$F_s(x) L_{s(q+1)}(x) - L_s(x) F_{s(q+1)}(x) = 2(-1)^{s+1} F_{sq}(x),$$

and

$$(x^2 + 4) F_{sq}(x) F_{s(q+1)}(x) + L_{sq}(x) L_{s(q+1)}(x) = 2L_{s(2q+1)}(x).$$

We leave the details of the calculations to the reader. ■

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